Vol. 88, No. 4, October 2015

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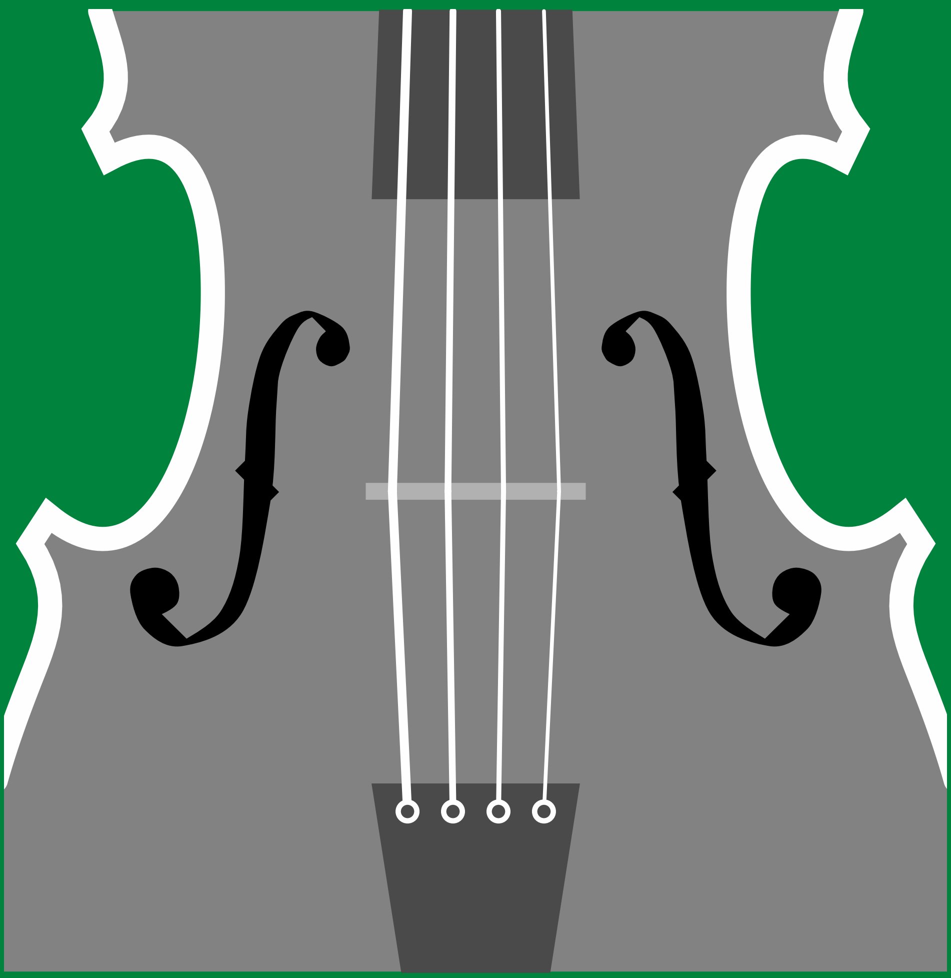
MATHEMATICS MAGAZINE



- Cubic splines and the shape of a violin
- Möbius maps, continued fractions, and a result of Galois
- Constructing approximate *n*-gons with straightedge and compass
- Combinatorial approach to powers of 2×2 matrices

EDITORIAL POLICY

Mathematics Magazine aims to provide lively and appealing mathematical exposition. The *Magazine* is not a research journal, so the terse style appropriate for such a journal (lemma-theorem-proof-corollary) is not appropriate for the Magazine. Articles should include examples, applications, historical background, and illustrations, where appropriate. They should be attractive and accessible to undergraduates and would, ideally, be helpful in supplementing undergraduate courses or in stimulating student investigations. Manuscripts on history are especially welcome, as are those showing relationships among various branches of mathematics and between mathematics and other disciplines.

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MATHEMATICS MAGAZINE (ISSN 0025-570X) is published by the Mathematical Association of America at 1529 Eighteenth Street, NW, Washington, DC 20036 and Lancaster, PA, in the months of February, April, June, October, and December.

Microfilmed issues may be obtained from University Microfilms International, Serials Bid Coordinator, 300 North Zeeb Road, Ann Arbor, MI 48106.

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Postmaster: Send address changes to Membership/Subscriptions Department, Mathematical Association of America, 1529 Eighteenth Street, NW, Washington, DC 20036-1385.

Printed in the United States of America.

COVER IMAGE

Middle of the Fiddle © 2015 David A. Reimann (*Albion College*). Used by permission.

The article "On the Shape of a Violin" by R. J. Stroeker was the inspiration for this piece. The author's discussion of clamped cubic splines informed the creation of the instrument's white border and f-holes.

Vol. 88, No. 4, October 2015

MATHEMATICS MAGAZINE

LETTER FROM THE EDITOR

Using cubic splines, Roel Stroeker gives a mathematical description of the backplate of a violin in the lead article of this issue. His article inspired David Reimann's cover art for the issue. The second article shifts from the construction of violins to the construction of n-gons. Get your compass and straight edge ready to follow Robert Milnikel's examination of the draftsman's construction, a way to approximate n-gons that was taught to engineers in the 1800's and early 1900's. Using trigonometric functions and their approximations, Milnikel is able to perturb the construction to achieve better outcomes.

Alan Beardon recognizes a relationship between one of my favorite topics (Möbius maps) and continued fractions. He uses the relationship to prove a theorem by Galois. Staying with more discrete ideas, John Konvalina offers a purely combinatorial formula for the nth power of a 2×2 matrix. The result applies properties of binomial coefficients in part by applying another favorite topic of mine: counting the number of positive integer solutions to an equality. The formula is used to prove two combinatorial identities; one of these identities involves Fibonacci numbers.

Mixed in is a proof without words from Roger Nelsen. This issue also includes a second anniversary crossword (Anniversary Crossword: Editors Past and President, Part II), the Problems section (with new problems, solutions, and quickies), and the Reviews section (with a review of two books on Benford's law).

News and Letters round out the issue. These report on the 44th USA Mathematical Olympiad and 6th USA Junior Mathematical Olympiad, the 56th International Mathematical Olympiad, and the 2015 Allendoerfer Awards for excellence in expository writing in the MAGAZINE.

Michael A. Jones, Editor

ARTICLES

On the Shape of a Violin

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My mother was a professional musician, and I loved listening to her playing the piano for hours on end. We, the children, too had to choose an instrument for our musical education, and at the age of eight, I began to study the cello. It took a while before I got the hang of it, but the practicing paid off and I began to really enjoy playing. I also liked being invited to join the school orchestra and several chamber music groups, and I became aware of the emotional impact the sound of a dead piece of wood could have on people. I promised myself that, given the chance, I would learn more about these miraculous stringed instruments and how they are made.

I became a mathematician, and I spend the next 40 years teaching and thinking about mathematics. Then after I retired, my sister and I happened to pass by the shop of a violin maker—a "luthier" to use the right word. She—the luthier—was very helpful, and I went on to a serious course in instrument making, building classical instruments in the tradition of the great masters of the past. I recently finished a viola da gamba, but I still have a long way to go in unraveling the secret of its magic sound.

This article is about one way in which mathematics can be applied to the art of lutherie. It addresses this particular problem: How can we give a mathematical description of the backplate of the violin? We limit ourselves to its first appearance as a two-dimensional piece of wood.

The overall shape of instruments of the violin family (violin, viola, cello) has changed very little over the past hundreds of years, in contrast to that of most other stringed instruments. So, this shape can certainly be seen as rather successful, possibly even ideal in the sense of most natural or most visually pleasing. We will use the mathematics of cubic splines, in particular parametrized cubic splines. The main reason for choosing cubic splines lies in their unique curvature properties, which might help us in our attempt to give an explanation for this ideal shape. These techniques may become very helpful to luthiers. But our main purpose here is to describe them to mathematicians and to demonstrate their strengths in the context of this wonderful application.

The backplate

Studying the ways in which stringed instruments are constructed, I learned that, on the one hand, it is common practice to copy famous instruments in minute detail, on the

Math. Mag. **88** (2015) 247–259. doi:10.4169/math.mag.88.4.247.
MSC: Primary 65D07, Secondary 65D10; 41A15.

other hand, if a new model is desired, with a few exceptions, only marked ruler and compass are used in the construction. Often, the ways in which such constructions are laid out are complicated and appear to be rather ad hoc (see [**1**, Tafel I]). "We need a round curve here, so which circle serves our purpose best," seems to be the adage, and where circles meet, the sharp intersection is generally smoothed over. An exception is the catenary, a curve defined by the formula

$$y = a\cosh(x/a) = \frac{a}{2}\left(e^{x/a} + e^{-x/a}\right) \text{ with } a > 0,$$

which sometimes helps to shape the arching of the plates of the violin. The constant a can be expressed in terms of the length of the centerline of the backplate. But generally mathematical formulae are shunned.

The measurements required for the construction sometimes follow certain patterns, like those based on the golden section (see [**5**]). Often, they seem to come from local considerations and follow no general rule or philosophy.

Figure 1 Template (left) and backplate (right) of base model after Antonio Stradivari 1689.

When starting on a new violin, it is common practice to first make a template that serves as a model for the mould to which the sides of the violin (the ribs) have to be glued (see FIGURE 1, left). The ribs are approximately 1.2 mm thick, and both front and back plates protrude from the sides by another 2.5 mm so that the outer form of the violin is slightly larger, which is especially noticeable at the points where the upper and lower parts change into the Cs (see FIGURE 1, right). Here, we are looking at the two-dimensional form of the soundbox of a violin. At this point, I would like to draw your attention to the wonderful treatise on the *Art of Violin Making* by Chris Johnson and Roy Courtnall [**8**].

When designing a new violin, it is the template that we have to construct first. However, since we wish to compare our construction to actual instruments, we shall instead focus on the outer form. We choose the backplate as the best representative of this form, and so we ignore the neck and the scroll. Moreover, the measurements we shall use are those of a Stradivari violin built in 1689; from here on, we shall refer to this instrument as the *base model**; see FIGURE 1. A list of 115 points from which the outer form of the backplate of this model can be constructed is available at [**16**].

Observe that the backplate is symmetrical, so it is enough to describe the curve on one side. The piecewise smooth curve in FIGURE 2 consists of five pieces, the lower curve $L_1 \ldots L_6$, the upper curve $U_1 \ldots U_6$, the C-curve $C_1 \ldots C_7$, and two short line segments connecting the lower curve with the C-curve and the upper curve with the C-curve, L_6C_1 and U_1C_7. All these curves lie in the same plane. The positions of the L, C, and U points will be clarified in due course.

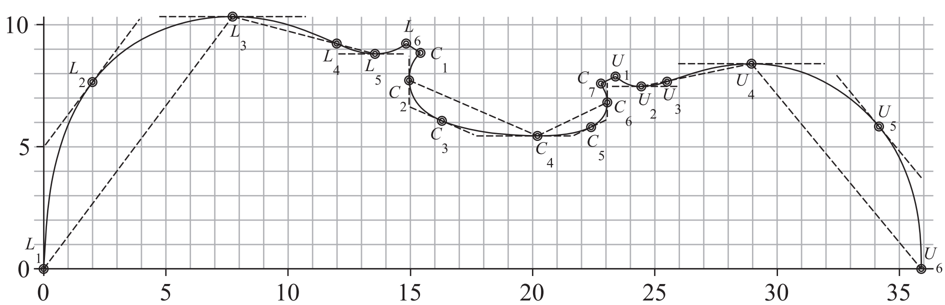


Figure 2 Significant points on the base model's outer form with construction lines.

It is obvious that starting the construction we have to set off with a number of given measurements. These roughly determine the outer form of the instrument. Initially, we have to decide on the length of the center line, which divides the back in two symmetric halves. It is such a symmetric half we are interested in. We also have to know the position and size of the C-parts and the largest and smallest width on the upper and lower parts. All this is necessary so that the final result could rightly be called a violin. This also means that, although we may choose our own measurements, we must not loose sight of the fact that the margins are rather small. Typical measurements of a standard violin are given in TABLE 1.

TABLE 1: Standard backplate measurements.

Standard violin	mm
length	356
upper width	168
middle width	112
lower width	208

The measurements vary only slightly with a variation of the length of the body of at most 10 mm. Useful information on measurements is given in [**15**].

Next, we choose a number of points on the outline of the backplate in agreement with the measurements we set out with. We shall refer to these points as *guide points*. There are a few obvious ones, like the endpoints of the center line U_6 and L_1 and the other endpoints of upper and lower parts (U_1, L_6) and of the C-part (C_1, C_7) and maybe a few others, corresponding to maximal width (U_4, L_3) and minimal width (C_4) for instance (see FIGURE 2). We shall continue the discussion on the choice of guide points, after first looking at cubic splines.

*This model is in use in the violin class at the CMB (Centrum voor MuziekinstrumentenBouw); see [**4**].

Cubic splines, the basics

Now let us state our mathematical problem.

We are given a sequence of distinct guide points (x_i, y_i), $i = 0, 1, \ldots, n$, in the plane, and we are seeking to find a "nice" plane curve passing through these points in the order given by $i = 0, 1, \ldots n$. Here, "nice" means smooth, visually attractive, and with no unnecessary bending.

Let us first assume these guide points are placed in such a way that the graph of a proper function s can pass through them. Later on, we will extend this to the case of points that are placed arbitrarily. Then, because the graph of a function can obviously not turn on itself, the x-values (the *knots*) must be ordered like $a := x_0 < x_1 < \cdots < x_n =: b$. So $y_i = s(x_i)$ for each i. Clearly, this is an interpolation problem, so let us consider polynomial interpolation. Because high degree polynomial interpolation usually comes with many oscillations, we should go for piecewise low degree polynomial interpolation. Piecewise linear interpolation is not smooth at the knots, and piecewise quadratic polynomial interpolation does not give us enough freedom to control the smoothness at the knots. Therefore, we choose piecewise cubic interpolation.

The function $s : [a, b] \to \mathbb{R}$ is called a *cubic spline* if it satisfies the following conditions:

1. $s = s_i$ is a cubic polynomial on the subinterval $[x_{i-1}, x_i]$ of [a,b] for $i = 1, \ldots, n$,
2. $s(x_i) = y_i$ for $i = 0, \ldots, n$,
3. $s_i^{(j)}(x_i) = s_{i+1}^{(j)}(x_i)$ for $i = 1, \ldots, n - 1$ and $j = 0, 1, 2$.

The first condition tells us that s is a piecewise cubic polynomial on $[a, b]$, and the second one says that s is an interpolation function on the set of guide points. The third condition expresses the smoothness of s at the knots. It says $s \in C^2[a, b]$, which means that s is a twice continuously differentiable function on the closed interval $[a, b]$. The s_i are cubic polynomials, and therefore, s can be explicitly given by $4n$ coefficients. On the other hand, the interpolation and smoothness conditions amount to a total of $n + 1 + 3(n - 1) = 4n - 2$ equations. Therefore, we may impose two extra boundary conditions. A natural choice is $s''(a) = s''(b) = 0$, which gives the so-called *natural cubic spline*. Another choice is to fix the vector $(s'(a), s'(b))$, and this is known as the *clamped cubic spline*. The cubic spline function s is uniquely determined by the three conditions plus the two boundary conditions. In the next section, we shall give a proof of this by construction.

Now, the curves we need for our purpose cannot always be given by the graphs of spline *functions*. This can clearly be seen in FIGURE 2: The C-curve, when run through from left to right, turns on itself, and is therefore not the graph of a function. That is why we need so-called *parametric splines*. We understand a parametric spline to be a plane curve given by the set of points

$$\{(x(t), y(t)) : a \leq t \leq b\},$$

where x and y are spline functions of the parameter t. Often, one chooses $a = 0$ and $b = 1$. How should the parametrization be chosen? This is an important point. The interpolation points are generally not uniformly spaced, and therefore, different parametrizations should give different splines. Also, the curves we are after are certainly nonsingular, so our parametrized curves should also be nonsingular. The most natural choice is to take t to be the arc length of the curve. But it is almost always rather difficult to find an explicit expression for the arc length of a given curve, and our curves are no exception. We should also take into account the fact that successive guide points are not placed at equal distances. So, if for $i = 0, 1, \ldots, n$ the points

$A_i = (x_i, y_i)$ are the guide points, then we define $l_0 := 0$, $l_{i+1} := l_i + \|A_{i+1} - A_i\|_2$ for $i = 0, 1, \ldots, n-1$ so that l_n is the sum of the line segments joining successive guide points, which is then the piecewise linear approximation of the arc length between A_0 and A_n. Then choose $t_i := l_i / l_n$ so that $t_0 = 0$ and $t_n = 1$. Thus, $t_0, t_1, \ldots, t_n$ are the knots of the spline functions $x(t)$ and $y(t)$, the components of the parametric spline $(x(t), y(t))$ with $0 \leq t \leq 1$. Further, $x_i = x(t_i)$ and $y_i = y(t_i)$ for all i.

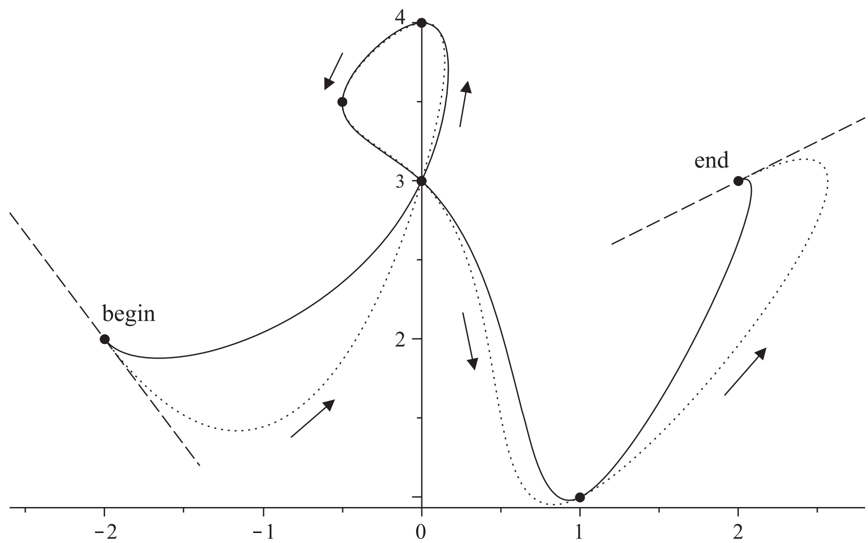


Figure 3 Parametric cubic splines through the same points and with equal end conditions but with different multiplication factors at the endpoints.

Dealing with clamped cubic spline functions, we need to choose the tangent directions at both endpoints to make the splines unique. With parametric clamped cubic splines, we have even more freedom. Indeed, let the curve be given by $f(x, y) = 0$ with a parametrization as given above. Traversing the curve from $t = t_0$, let us set off in the direction of the vector α with $\|\alpha\|_2 = 1$. This means $(x'(t_0), y'(t_0)) = \alpha$ because $[x'(t_0), y'(t_0)]$ is the direction of the tangent to the curve at A_0. However, this fixes neither $x'(t_0)$ nor $y'(t_0)$—required for clamped cubic splines—because $m \cdot \alpha$ gives the same direction for all real $m \neq 0$. This can also be seen as follows. As the curve is nonsingular, either $x'(t_0) \neq 0$ or $y'(t_0) \neq 0$. Without loss of generality, we assume $x'(t_0) \neq 0$. It then follows, that in a neighborhood of A_0, the curve can be given by $y = \phi(x)$ for a differentiable function ϕ. Now let $x'(t_0) = m \cdot \alpha_1$ and $y'(t_0) = m \cdot \alpha_2$, then $\phi'(x(t_0)) = y'(t_0)/x'(t_0) = \alpha_2/\alpha_1$, and the factor m drops out. We shall call this factor m the *multiplication factor*. So the value we choose for m does not affect the direction of the tangent at A_0. Changing the multiplication factor does not alter the tangent, but the larger m, the closer the graph of the curve is drawn toward the tangent. Naturally, this also applies to the other endpoint A_n. We shall always take the multiplication factor positive. We therefore may have to alter the sign of the direction of the tangent, depending on the way we traverse the curve. In FIGURE 3, the multiplication factor of the dotted spline is much larger than that of the black spline. The directions of the tangents (the dashed lines) at the endpoints are $[3, -4]$ and $[-2, -1]$, respectively.

Cubic splines, advanced properties

As we saw in the previous section, a cubic spline determines $4n-2$ equations in $4n$ unknown coefficients. The existence and uniqueness of natural and clamped cubic splines can be shown by construction.

The function s'' is a piecewise linear polynomial. Write $\sigma_i = s''(x_i)$ for $i = 0, 1, \ldots, n$. Then for $i = 1, \ldots, n-1$ we have

$$s''(x) = \begin{cases} \frac{x-x_i}{x_{i+1}-x_i}\sigma_{i+1} + \frac{x_{i+1}-x}{x_{i+1}-x_i}\sigma_i & \text{for } x_i \le x \le x_{i+1} \\ \frac{x-x_{i-1}}{x_i-x_{i-1}}\sigma_i + \frac{x_i-x}{x_i-x_{i-1}}\sigma_{i-1} & \text{for } x_{i-1} \le x \le x_i \end{cases},$$

because s'' is a linear function. Working out the repeated integrals

$$\int_{x_i}^{x_{i+1}} \int_{x_i}^{x} s''(t)\,dt\,dx \quad \text{and} \quad \int_{x_{i-1}}^{x_i} \int_{x}^{x_i} s''(t)\,dt\,dx$$

in two ways and eliminating $s'(x_i)$ from the resulting equations eventually leads to the relations

$$\omega_i\sigma_{i-1} + 2\sigma_i + (1-\omega_i)\sigma_{i+1} = r_i \quad \text{for } i = 1, \ldots, n-1, \tag{1}$$

where

$$\omega_i := \frac{x_i - x_{i-1}}{x_{i+1} - x_{i-1}} \quad \text{and} \quad r_i := \frac{6}{x_{i+1} - x_{i-1}}\left(\frac{y_{i+1} - y_i}{x_{i+1} - x_i} - \frac{y_i - y_{i-1}}{x_i - x_{i-1}}\right).$$

Combining relations (1) into a single matrix equation, for the natural boundary conditions, we get the linear system

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \omega_1 & 2 & 1-\omega_1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \omega_2 & 2 & 1-\omega_2 & & 0 & 0 & 0 \\ 0 & 0 & \omega_3 & 2 & \ddots & & & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \cdots & \omega_{n-1} & 2 & 1-\omega_{n-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sigma_0 \\ \sigma_1 \\ \vdots \\ \sigma_{n-1} \\ \sigma_n \end{pmatrix} = \begin{pmatrix} 0 \\ r_1 \\ \vdots \\ r_{n-1} \\ 0 \end{pmatrix}. \tag{2}$$

The coefficient matrix of (2) is tridiagonal and strictly diagonally dominant because $0 < \omega_i < 1$ for each i. This means that this system is uniquely solvable (see [**7**, Theorem 6.1.10 on page 349]), which implies that the natural cubic spline exists and is unique. It also gives us the values of $s''(x)$ at the knots. Tracing back down the intermediate relations gives us the coefficients of each s_i. In the case of the clamped cubic spline, we find a slightly more complicated matrix equation, but the conclusions remain the same. For both sets of boundary conditions and for any function $f \in C^m[a, b]$ with $m \ge 2$ for which $y_i = f(x_i)$ for each i, and in addition $f' = s'$ at both endpoints for a clamped cubic spline—which means coinciding tangents at the endpoints—we can show the inequality

$$\int_a^b [f''(x)]^2\,dx \ge \int_a^b [s''(x)]^2\,dx, \tag{3}$$

where either $(s'(a), s'(b)) = \alpha$ or $s''(a) = s''(b) = 0$. The proof runs as follows (see [**11**, Ch. 4]. Consider

$$\int_a^b [f''(x) - s''(x)]^2\,dx + \int_a^b [s''(x)]^2\,dx = \int_a^b [f''(x)]^2\,dx - 2\int_a^b s''(x)[f''(x) - s''(x)]\,dx.$$

Using integration by parts on the last integral on the right gives

$$\int_a^b s''(x)[f''(x) - s''(x)]\,dx = s''(x)[f'(x) - s'(x)]\Big|_a^b - \int_a^b s'''(x)[f'(x) - s'(x)]\,dx.$$

The first term on the right vanishes because of the boundary conditions and so does the second term in view of the fact that s''' is constant on each subinterval (x_{i-1}, x_i). Indeed, if $s'''(x) = c_i$ for $x \in (x_{i-1}, x_i)$ and $i = 1, \ldots, n$, then

$$\int_a^b s'''(x)[f'(x) - s'(x)]\,dx = \sum_{i=1}^n c_i \int_{x_{i-1}}^{x_i} [f'(x) - s'(x)]\,dx = 0.$$

Inequality (3) has an important geometric interpretation that partly explains the reason for the popularity of the cubic spline. The mathematical *curvature* $\kappa(x)$ of a twice continuously differentiable function $f : [a, b] \to \mathbb{R}$ at the point $x \in [a, b]$ is defined by the formula

$$\kappa(x) = \frac{f''(x)}{(1 + [f'(x)]^2)^{\frac{3}{2}}}. \tag{4}$$

The curvature of a circular arc of radius R is $1/R$ or $-1/R$, depending on the orientation of the circle. Assuming $|f'(x)| \ll 1$ on $[a, b]$—admittedly, this is not always the case—we see that the norm $\| \kappa \|_2^2$ is approximately equal to $\int_a^b [f''(x)]^2\,dx$ so that inequality (3) now says that of all the $C^m[a, b]$ functions with $m \geq 2$ and satisfying the interpolation conditions, *the cubic spline has the smallest total curvature in the sense of the* ℓ_2*-norm.* Of course, $\int_a^b [f''(x)]^2\,dx$ merely gives a coarse measure of the total curvature.

There is yet another interpretation of inequality (3), and this one explains the reason for the name "spline" that is given to this interpolation function. Consider a thin homogeneous isotropic flexible rod whose center line is given by a function $f : [a, b] \to \mathbb{R}$. Then the total bending energy is given by the formula

$$c\int_a^b \frac{[f''(x)]^2}{(1 + [f'(x)]^2)^3}\,dx \approx c\int_a^b [f''(x)]^2\,dx$$

for a certain constant c and under the assumption $|f'(x)| \ll 1$ on $[a, b]$. If such a rod is forced to go through a number of fixed points, in such a way that only forces perpendicular to the rod are applied, it assumes a position of minimal energy. Therefore, inequality (3) now asserts that the center line of this rod approximately follows the natural cubic spline through these points. Outside of the interval $[a, b]$, no force is applied to the rod, and therefore, it assumes the natural shape of a straight line. In that sense, the boundary conditions $s''(a) = s''(b) = 0$ should be seen as "natural." It now makes sense why this type of interpolation function was given the name "spline" because a mechanical spline is a thin flexible rod that is used by draughtsmen (e.g., in shipbuilding) for drawing smooth curves through a number of fixed points. It was I. J. Schoenberg who introduced the name "spline" in 1946 (see [**12**]). See also the foreword by A. Robin Forrest in [**2**].

Cubic splines are by far the most popular of all spline functions; they are especially useful for approximation purposes. In particular, every function continuous on a closed interval $[a, b]$ can be arbitrarily well approximated on $[a, b]$ by cubic splines provided sufficiently many knots are available. In fact, if s is a cubic spline that interpolates $f \in C^m[a, b]$ at the knots $a = x_0 < x_1 < \cdots < x_n = b$, then

$$\| s - f \|_\infty = O(h^{m+1}) \text{ as } h \downarrow 0, \text{ where } m = 1, 2, 3 \text{ and } h := \max_{i=1,\ldots,n} (x_i - x_{i-1}).$$

See [**6**] and [**17**, 9.7.3]. General treatises on splines are [**3**] and [**14**].

The construction

Now, we are ready to construct the "ideal" outline of the backplate with as few guide points as possible, where ideal is meant in the sense of "visually pleasing" with no unnecessary bending. Of course, avoiding unnecessary bending goes hand in hand with avoiding the use of too many guide points so that the natural curving property of splines is not thwarted.

Naturally, we want to compare the outline of the base model with the approximation obtained by means of parametric splines with a given small set of guide points, the *spline model*. It turns out that, with the guide points chosen in accordance with the measurements of the base model (all 115 data points), together with the approximate data at the endpoints (see TABLE 2 and the paragraph immediately following FIGURE 4), we get an outline that compares very favorably with that of the actual instrument. From now on, we shall call this outline also the *base model* or the *base contour*.

TABLE 2: Tangent directions with multiplication factors at endpoints in FIGURE 4.

Guide point (endpoint)	Direction tangent	Multiplication factor	
		8-point spline	11-point spline
L_1	$[0, 1]$	32	34.5
L_6	$[1.5, 1]$	20	28
C_1	$[-1, -1.2]$	14.6	13.1
C_7	$[-1, 1.8]$	13.5	16.2
U_1	$[1.2, -1]$	17	21
U_6	$[0, -1]$	24	25

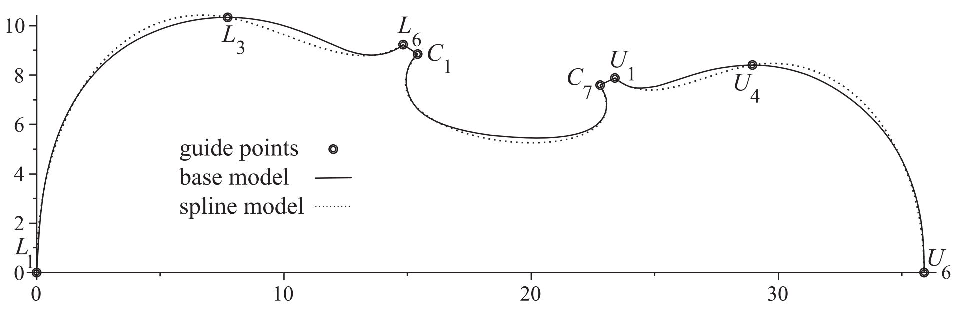


Figure 4 8-point spline compared with the base model.

Choosing the number of guide points and the points themselves is crucial. First, we should not choose too many points for fear of designing the contour ourselves, instead of leaving this to the spline engine. Further, the choice of guide points should be mainly imposed by necessity because of the obvious requirements of size and shape.

The procedure is now as follows. We consider the backplate of our model, or rather half of it, and we try to find points on its contour at the most significant positions. Nineteen possible candidates with an indication on their construction are shown in FIGURE 2. Four groups of points can be distinguished: endpoints (L_1, L_6, C_1, C_7, U_1, U_6), points at extremal positions (L_3, L_5, C_2, C_4, C_6, U_2, U_4), points with tangents parallel to line segments (L_2, C_3, C_5, U_5), and points at the intersection of line segments and the base model (L_4, U_3). From this set of points, we shall choose a subset, the guide points. And finally, these guide points will serve as interpolation points for our parametric cubic splines. We shall also compare the result with the base model.

For each of the parts L, C, U, we shall construct a parametric cubic spline. First, we need to decide on the position of the endpoints of these three parts, namely L_1 and L_6, C_1 and C_7, U_1 and U_6. There cannot be much doubt about the necessity of the choice of these points as guide points, but clearly, this choice cannot possibly be sufficient; at least two more guide points are needed to get anything like the correct shape, typical for a violin. We also must choose the direction of the tangents at these six endpoints and their multiplication factors. An important point is this: We can choose the multiplication factors to suit us best, which could either mean so that we like the resulting curve best or that it matches the base model best. The tangents at L_1 and U_6 must be vertical, and the tangents at the other four endpoints can be freely chosen, say to run at an angle anywhere between 30 to 60 degrees in a positive or negative sense. The upper and lower parts are quite similar, so let us consider the lower part. Provided we choose a point between L_1 and L_6 with an ordinate value larger than that of L_6, the resulting parametric spline through these three points will have the right sort of shape. This is also true for the upper part; the C-part, however, does not need another point. The extra points that certainly satisfy the restrictions are the highest points on the lower and upper parts, namely L_3, and U_4. In FIGURE 4 we compare the shape generated by the cubic parametric splines through these eight guide points with that of the base model. See also TABLE 2 for the tangents and the multiplication factors. The latter are chosen such that the spline curves at the endpoints optimally agree with those of the base model.

To be able to compare our spline model with the base model, we need to know the tangent directions at the endpoints of the base model. Clearly, at the points L_1 and U_6, the exact tangents are known because of symmetry considerations. At the other four endpoints, we do not know the exact tangent directions, but it is not difficult to obtain good approximations. At each of these four points, we have simply taken the direction of the line segment connecting this endpoint with its neighboring point. See TABLE 2 for the result. Next, take point L_1. Choose the multiplication factor by observation in such a way that the lower part of the spline model coincides with the base model as much as possible: Increasing a multiplication factor draws the curve closer toward the tangent. A similar action applies to U_6. The remaining four points have to be treated slightly differently because we do not have the exact tangent directions of the base model at these points. Here, we also have to vary the tangent directions of the base model slightly to obtain a visually optimal situation. Of course, these solutions are not mathematically exact, but that is not the point here. We are mainly interested in the optimal bending properties of cubic splines. Thus, trying to determine the best multiplication factor mathematically is not really important because *any* choice of this factor gives a spline with optimal bending and thus a beautiful model, but here we want to see how well we can cover the base contour!

Although the shape of the spline curve in FIGURE 4 seems quite acceptable, there is also room for improvement. Observe that the points L_3 and U_4 are not really close

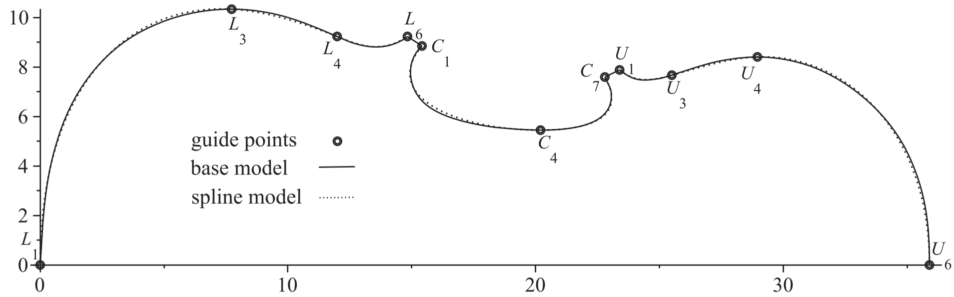


Figure 5 11-point spline compared with the base model.

to extremal positions on the spline and that the C-part spline curve possibly makes the waist too small. So if one wishes to stay closer to the base model, extra points are needed, like L_4, C_4, and U_3. The points L_4 and U_3 have the effect of "flattening off" the splines toward the base model. Of course, any points between L_3 and L_5 and between U_2 and U_4 will have that same effect. Observe that in TABLE 2 the directions of the tangents are the same, but the multiplication factors are not. Although not perfect, the match of the 11-point spline (see FIGURE 5) is quite good. Other choices of guide points can be made, and a perfect match can be easily obtained by taking a few extra points from the 19 significant points of FIGURE 2.

Curvature

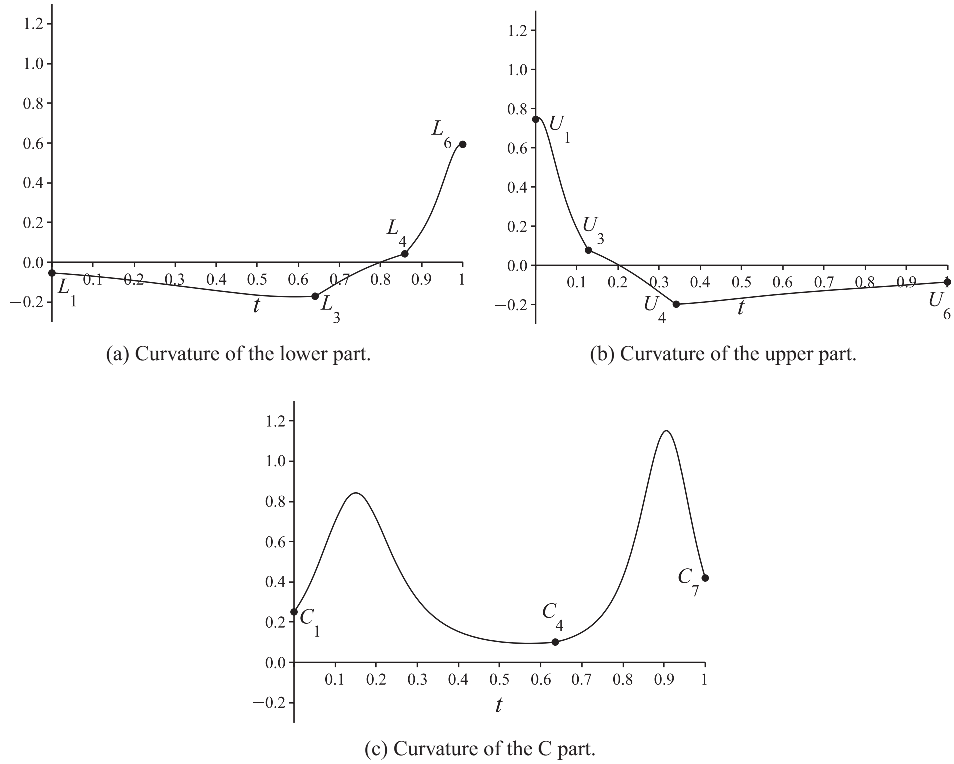
We have seen that the combined spline contours through the 11 given guide points provide a reasonably close fit for the base model. Another interesting point that can be made is about the curvature: How does the curvature of the base model change from point to point? Can the answer throw light on the way luthiers in their marked ruler and compass constructions use circular arcs?

The curvature of a twice continuously differentiable function at each point of its graph is given by equation (4). From this, the formula for the curvature of a parametric curve $C = \{(x(t), y(t)) : a \leq t \leq b\}$ at each value of the parameter t can be deduced easily:

$$\kappa(t) = \frac{x'y'' - x''y'}{[(x')^2 + (y')^2]^{\frac{3}{2}}}. \tag{5}$$

For convenience, we have dropped the t in the right-hand side of formula (5). In order to check this formula, suppose $x'(t) \neq 0$ at the parameter value $t = t_0$. Then the curve C may be given in a neighborhood of t_0 by an equation $y(t) = \phi(x(t))$ for a twice continuously differentiable function ϕ. Then dropping the t again, $y' = \phi'(x)x'$ and $y'' = \phi''(x)(x')^2 + f'(x)x''$, from which in combination with (4) the assertion easily follows. The curves we consider have no singular points. So if $x'(t) = 0$ at $t = t_0$, then certainly $y'(t_0) \neq 0$, and a similar argument may be given. Recall that the circle of curvature at point $t = t_0$ has radius $1/\kappa(t_0)$. It is the circle whose center lies on the normal to the curve at that point and whose curvature agrees with that of the curve at t_0.

It follows from formula (5) that the curvature function $\kappa(t)$ is continuous on the entire range $[t_0, \ldots, t_n]$. Nevertheless, the curvature of C^2-curves is rather sensitive to small changes. If our curve is a parametric cubic spline, it is unlikely that $\kappa(t)$ is differentiable at the knots. So our function $\kappa(t)$ will probably not have a very smooth appearance.



(a) Curvature of the lower part.

(b) Curvature of the upper part.

(c) Curvature of the C part.

Figure 6 Curvature of the parametric 11-point spline curves.

Recall that the Stradivari base model is obtained by careful observation of 115 successive data points about 5 mm apart. Joining these points by means of parametric cubic splines to obtain a close approximation of the model gives a good result, but the corresponding curvature function does not look so good as a result of the phenomenon mentioned above.

Fortunately, the 11-point spline approximation of the base contour is also quite good, and what is more, its curvature function is much smoother. It is therefore preferable to consider the curvature function of our parametric spline approximation. In FIGURE 6, the curvature functions of the lower, C-, and upper parts are shown. It appears that the curvature at both ends L_1 and U_6 is almost constant and small. Although one should be very careful with one's interpretation of this, it could mean that the contour at both L_1 and L_6 is similar to a circular arc. TABLE 3 shows the approximate radius of the curvature circle at each of the guide points. We also observe in FIGURE 6(c) that the curvature of the C-part is rather regular; around the center the curvature is almost constant, which might indicate that the major curve resembles a circle with radius of approximately 9.9 cm. Also, the small circular arcs near the endpoints are clearly visible.

Note on the practical application

So far we have considered the mathematical definition and properties of parametric cubic splines. But how can these be handled in practice? All calculations done so far, including all graphs, have been created by means of the computer algebra package Maple (see [**9**]). But in order to use splines, one does not need intimate knowledge

TABLE 3: Coordinates (x, y) with parameter value $t \in [0, 1]$ of the 11 guide points (see FIGURE 5) with their curvature $\kappa(t)$ and radius R of the curvature circle. Coordinates and radius are measured in centimeters

Guide point	$x(t)$	$y(t)$	t	$\kappa(t)$	R
L_1	0.0	0.0	0	-0.0538253196	18.57861703
L_3	7.7279227	10.3352606	0.6406923777	-0.1707587319	5.85621590
L_4	11.9861762	9.2333787	0.8590647440	0.0425922197	23.47846643
L_6	14.8249271	9.2336330	1	0.5932583552	1.68560627
C_1	15.4128687	8.8493857	0	0.2506912160	3.98897104
C_4	20.1975232	5.4481232	0.6353508310	0.1009567893	9.90522784
C_7	22.7944348	7.5946695	1	0.4193279880	2.38476808
U_1	23.3900054	7.8817742	0	0.7454555014	1.34146169
U_3	25.4994239	7.6701966	0.1280534854	0.0777210168	12.86653264
U_4	28.9525636	8.4071580	0.3413289745	-0.1978771462	5.05364070
U_6	35.8974966	0.0	1	-0.0845792332	11.82323322

of a computational nature. It is sufficient to know how to work with a CAD system, like VectorWorks, TurboCad, or AutoCAD, to name but a few. These systems use vector graphics and they implement parametric cubic splines through B-splines (B stands for basis). These systems are mainly used for experimental design purposes, where curves are designed and used interactively. Now, cubic splines—at least in the way presented here—are less useful for experimental purposes because changing even a single interpolation point changes the entire spline instead of the parts closest to the point changing so that all calculations have to be done all over again. B-splines cure that problem by constructing the spline locally between two successive interpolation points as a linear combination of (four, in case of cubic splines) basis splines. Further, in design purposes, there are generally no severe restrictions as to the points the curve has to pass through. With B-splines one uses so-called *control points* to change the shape of the curve, but the curve generally does not pass through these control points. So, mathematical splines are calculated by interpolation, and B-splines use a different computational technique based on the linear combination of basis splines. For those who wish to know more about the mathematical background of B-splines, Bézier curves and NURBS (nonuniform rational B-splines) we refer to [**13**] for a very nice and detailed overview. See also [**10**].

Conclusion

We have seen that by choosing guide points in a certain way a quite acceptable model for the backplate of a violin can be constructed with the nice curvature properties of cubic splines. Of course, the great masters would have had no need for these fancy methods—if they had known them—in their quest for the most suitable but also most gracious and beautiful shape, but we, ordinary mortals, might find the help modern methods can give us quite useful. These modern methods are accessible through CAD software. Even so, and fortunately too, in the search for suitable guide points, human intervention—in other words, the eye of the master—remains indispensable.

Acknowledgments I am grateful to Sholem van Collem, professional violinist and accomplished amateur luthier, who put me on the right track in my search for a suitable set of guide points. I also thank him for his assistance in obtaining precise measurements for the outer form of the base model and above all for many hours of in-depth discussion on the making of violins.

I also wish to express my thanks to the referees for their useful suggestions.

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Summary. For centuries luthiers—that is, instrument makers of violins and other stringed instruments—had no more sophisticated tools at their disposal to define the shape of their instruments than marked ruler and compass. Today, modern aids are available in terms of computational power and expertise in graphic design to assist them in this respect. This raises the following question: How can these powerful computational techniques be applied in the process of searching for a form of the violin both pleasing to the eye and optimal in some mathematical sense? In this paper, I use parametric cubic splines in an attempt to come close to and possibly improve upon—strictly in a mathematical and visual sense—the shape of a violin as laid down by the great masters of the past. The main reasons for choosing the cubic spline are: good approximation properties, simplicity of construction, and most importantly, its unique curvature properties.

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A New Angle on an Old Construction: Approximating Inscribed n-gons

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To the 21^{st}-century mathematics student compass-and-straightedge constructions—such as constructions of regular polygons—might constitute a fun puzzle, an entryway into the rich ideas of axiomatic geometry, or simply a relic of ages past. Before computer-assisted drafting technology was widely available, the compass and straightedge were vital tools for the engineer, the architect, and anyone else who needed precise drawings. There are innumerable sources for more background on compass-and-straightedge constructions, e.g., [**1**], [**10**], and [**6**].

Compass-and-straightedge constructions of some regular n-gons, such as for $n = 5$, 6, 8, and 10, were known in Euclid's time and likely well before. For others, such as $n = 7$ and 9, no exact construction was known, and for good reason. In the early 19^{th} century, Gauss conjectured in [**5**] and Pierre Wantzel proved in [**11**] that exact compass-and-straightedge constructions of regular n-gons are possible only if n is a power of 2 or a product of one or more Fermat primes times some power of 2. The only known Fermat primes are 3, 5, 17, 257, and 65537; see [**3**] for much more on these numbers.

The fact that no exact construction was known did not, however, keep draftsmen from having need of a method for drawing regular n-gons for values such as 7 and others. In this case, necessity was the mother of approximation, and engineers and draftsmen developed a single construction that would produce a reasonably accurate regular n-gon inscribed in a circle for any value of n between 4 and about 15. The accuracy degrades quickly as n increases toward 25, at which point the construction fails completely.

The draftsman's construction. We use $n = 7$ in the illustrations, but the same construction works for any $n \geq 4$. The first two steps are classical and can be found in Euclid's *Elements* ([**4**]); the details of how they are accomplished are not crucial to the rest of the construction. The procedure is:

1. Find a diameter of the given circle, with endpoints A and B ([**4**], Book III, prop. 1).
2. Divide the diameter into n equal segments $AD_1, D_1D_2, \ldots, D_{n-2}D_{n-1}, D_{n-1}B$ ([**4**], Book VI, prop. 9).
3. Use the compass to make two circular arcs with radius AB, one with center A, the other with center B. These arcs will meet at a point which we will call C directly below the center of the circle.
4. Draw a line segment connecting point C to D_{n-2}, the second mark from the right on the diameter, excluding the right endpoint. Let E_1 be the intersection of the circle and the line CD_{n-2} that is on the opposite side of AB from C. We now have two points of the n-gon, B and E_1. See Figure 1.

Math. Mag. **88** (2015) 260–269. doi:10.4169/math.mag.88.4.260.
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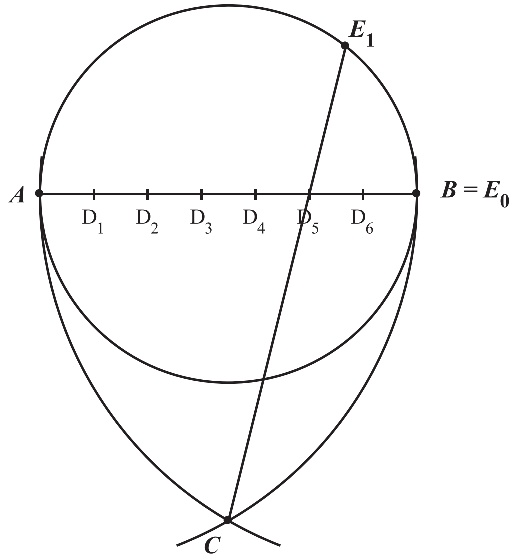

Figure 1 The draftsman's construction after stages 1–4, with $n = 7$. Note that D_5 is D_{n-2}.

5. Having found two points of the n-gon, we continue around the circle, using the distance between $B = E_0$ and E_1 as a guide. Successively construct points E_2, E_3, $\ldots$, E_{n-1} by constructing circles centered at E_i with radius $E_{i-1}E_i$. These smaller circles will intersect the large circle at points E_{i-1}, E_i, and E_{i+1}. See Figure 2.

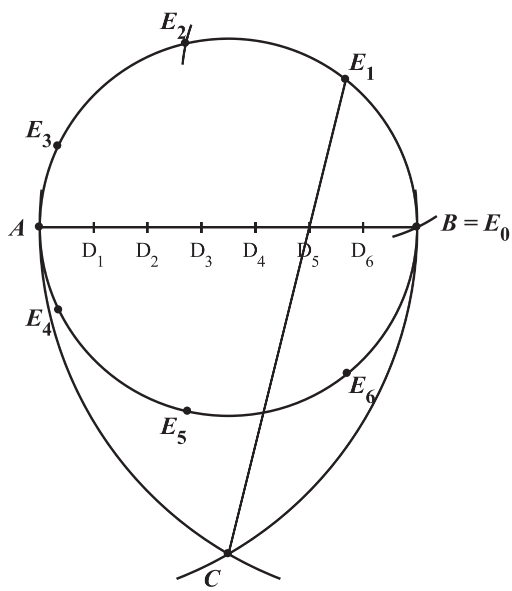

Figure 2 The draftsman's construction after stages 1–5, with $n = 7$.

6. The finished n-gon is $BE_1E_2 \cdots E_{n-1}$. Note that point A is typically not part of the n-gon.

This construction is not perfect. In the $n = 7$ case shown in the diagram, assuming a unit circle to begin with, the sides of the polygon should be about 0.8678 units long, and the construction produces side BE_1 that is 0.8692 units long, about 0.16% too large. All remaining sides except the last are of equal size, so this error accumulates, leaving the last side only 0.8593 units long, nearly 1% too short. This is the pattern for error in all n-gons ($n \geq 7$) made with this construction: The first $n-1$ sides are each a little long and the last side is too short by a factor of $n-1$ times the original error. For $n = 5$, it's the opposite, with four slightly short sides and one longer one; the construction is exact for $n = 4$ and $n = 6$.

This pattern of $n-1$ long sides and one short side becomes more and more obvious as we look at increasing values of n in Figure 3.

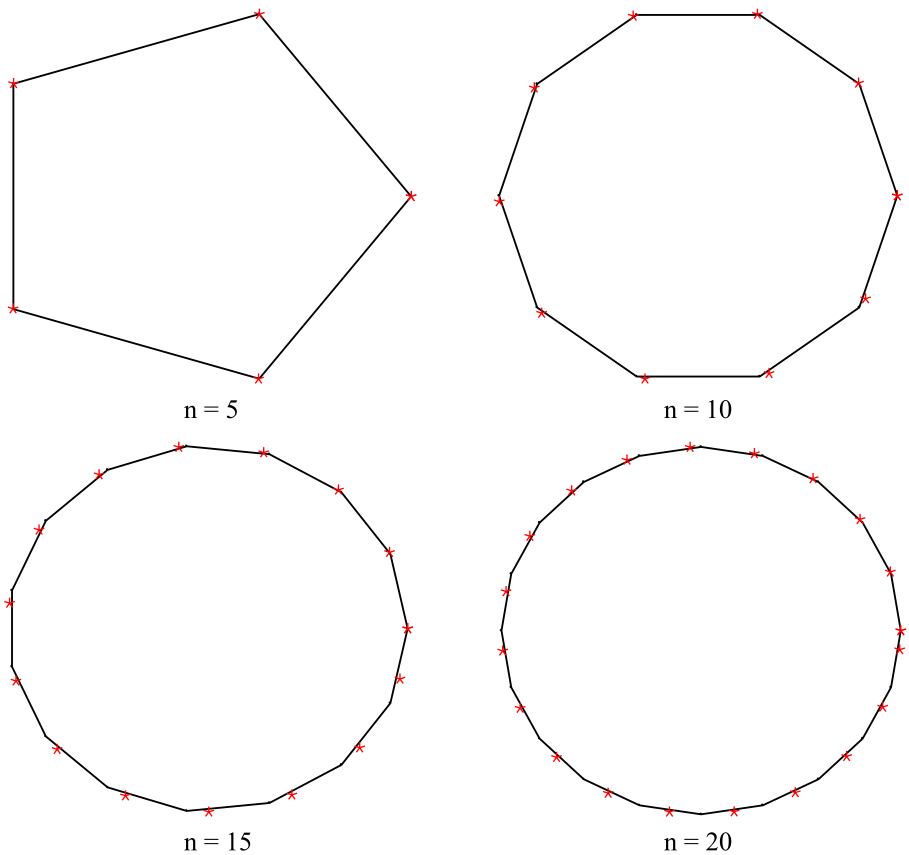

Figure 3 A regular n-gon is given in black, and the points generated by the draftsman's construction are marked with red asterisks. Notice that the pentagon is almost perfect, but the last side is noticeably short in the 15-gon and almost vanishingly short in the 20-gon.

We can now dissect the construction analytically to understand the patterns of error and see why it (nearly) works.

An algebraic analysis. Assume that we started with the standard unit circle, placing A at $(-1, 0)$ and B at $(1, 0)$; it is fairly straightforward to see that points C and D_{n-2} are at $(0, -\sqrt{3})$ and $(1 - \frac{4}{n}, 0)$, respectively. The placement of point E_1 in the construction determines the lengths of all of the sides of the inscribed n-gon.

Routine algebra tells us that the coordinates of E_1 are

$$\left(\frac{(3n+\sqrt{n^2+16n-32})(n-4)}{4(n^2-2n+4)}, \frac{\sqrt{3}(-n^2+8n-16+n\sqrt{n^2+16n-32})}{4(n^2-2n+4)}\right).$$

In a true n-gon, the coordinates of this point would be $(\cos(\frac{2\pi}{n}), \sin(\frac{2\pi}{n}))$, so the construction has provided us with approximations for $\cos(\frac{2\pi}{n})$ and $\sin(\frac{2\pi}{n})$. We can combine these to give an approximation of $\tan(\frac{2\pi}{n})$, and then take the inverse tangent to find the central angle the draftsman's construction produces for the first side of the n-gon. We find that

$$\frac{2\pi}{n} \approx \tan^{-1}\left(\frac{\sqrt{3}(-n^2+8n-16+n\sqrt{n^2+16n-32})}{(3n+\sqrt{n^2+16n-32})(n-4)}\right). \tag{1}$$

We now have a way to test the accuracy of the construction for, say, a 19-gon without actually making one. The important thing to measure here is not absolute error (comparing $\frac{2\pi}{19}$ to the approximation) but *relative* error, which is particularly important here, since the error will be compounded every time we add a side to the 19-gon. Table 1 shows the relative error in computing $\frac{2\pi}{n}$ for various values of n.

TABLE 1: Relative errors in constructing the first central angles. For n-values 25 and higher, the error is large enough that the construction does not produce an n-gon.

n	5	6	7	8	9	10
% Error in angle	-0.06	0	0.17	0.42	0.69	0.99
n	15	20	*25*	*30*	*60*	*360*
% Error in angle	2.40	3.52	4.39	5.07	7.21	9.67

These errors get quite large for large values of n. (For $n \geq 820$, they exceed 10%.) Moreover, these errors accumulate and are all paid for in the last side of the n-gon being too large or small. For example, the first 19 sides of a 20-gon accumulate error of $19 \cdot (3.52\%) \approx 66.9\%$, which means that the final side of a draftsman's construction 20-gon is only 33.1% as large as it should be. (See Figure 3.) When this error computation for the final angle yields a negative number, it is an indication that we have "wrapped around" before constructing all n sides and that the construction will not produce an n-gon at all. This happens for all $n \geq 25$. For example, in an attempted 60-gon, one actually works all the way around the circle by the time one has drawn only 56 sides.

The n for which the draftsman's construction ceases to be a "reasonable" approximation depends on the situation. For $n = 10$, the last angle is only about 10% too small—not great, but possibly tolerable, depending on the application. The last angle with $n = 7$ is about 1% too small, and the last angle in a pentagon is only about a quarter of a percent too large, virtually undetectable to the eye. The construction of the pentagon from Euclid's *Elements* (Book IV, Prop. 11), while exact, is extremely tedious compared to the draftsman's construction. (There are now known to be somewhat more efficient exact constructions, such as a lovely one by Herbert Richmond ([**8**]) from 1893.)

To better understand where the approximation in Equation (1) comes from and how it behaves, let's look at a continuous version of this approximation of the tangent function. By substituting $\frac{2}{x}$ for n and simplifying, we obtain

$$\tan(\pi x) \approx \frac{\sqrt{3}(-4x^2 + 4x - 1 + \sqrt{1 + 8x - 8x^2})}{(3 + \sqrt{1 + 8x - 8x^2})(1 - 2x)}. \tag{2}$$

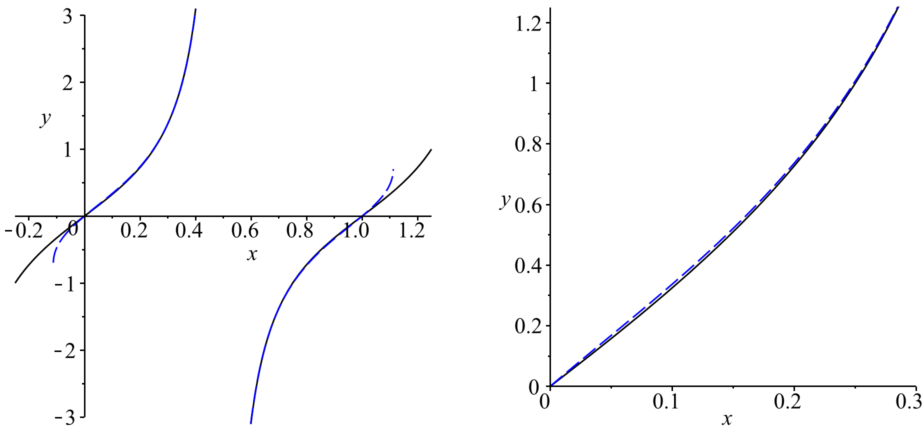

Figure 4 The function $\tan(\pi x)$ (solid black curve) and the approximation generated by the draftsman's construction (dashed blue curve) at two scales.

As we see in Figure 4, this approximation is reasonably accurate over the entire interval [0, 1], even though we need it only on the interval $(0, \frac{2}{5}]$. However, even small absolute errors in values near 0 can lead to large relative errors. If one looks at relative error in approximating the function $\tan(\pi x)$, the draftsman's approximation is quite good for values of x between about 0.2 and 0.5, but reaches over 10% error for values of x close to 0. This is exactly what we would expect from Table 1.

Hence, one reason the draftsman's construction is nearly accurate for small n and only modestly accurate for large n: Because the approximation to the tangent function we obtain is quite accurate near $x = 0.5$ and less so farther away. This leads naturally to the question of why the particular algebraic function we obtained is a good approximation to the tangent function in the first place.

An innovation

We have just seen the approximation from Equation (2) to be quite accurate for x near the asymptote at 0.5, but relatively less so for x near 0 and 1. However, we can reverse that fairly easily, moving the smallest errors over to $x = 0$. We will use the trigonometric identity $\tan\theta = -\cot(\theta + \frac{\pi}{2})$ and the fact that the tangent approximation (2) is symmetric around the asymptote $x = 0.5$.

As a first step, we begin with the negation of Equation (2), flip the numerator and denominator, and find that

$$-\cot(\pi x) \approx -\frac{(3 + \sqrt{1 + 8x - 8x^2})(1 - 2x)}{\sqrt{3}(-4x^2 + 4x - 1 + \sqrt{1 + 8x - 8x^2})}.$$

Now we shift this to the left by substituting $x + \frac{1}{2}$ for x, taking advantage of the fortuitous accuracy of the original approximation for values of x between 0.5 and 1. We find that

$$-\cot\left(\pi\left(x+\frac{1}{2}\right)\right) = \tan(\pi x) \approx \frac{2x(3+\sqrt{3-8x^2})}{\sqrt{3}(-4x^2+\sqrt{3-8x^2})} \tag{3}$$

after considerable simplification.

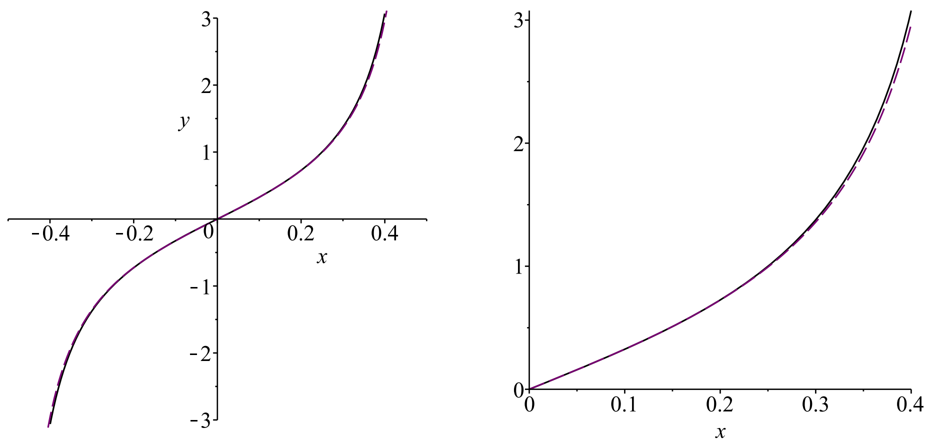

Figure 5 The function $\tan(\pi x)$ (solid black curve) and the new approximation from Equation (3) (dashed purple curve) at two scales.

Figure 5 confirms that this turns out to be a good approximation to the tangent function for values near $x = 0$, and still fairly close, though less accurate, as we approach the asymptote at $x = 0.5$. Even small absolute errors in values of $\tan(\pi x)$ near $x = 0$ might be large in relative terms, but the shifted approximation turns out to be excellent near $x = 0$, with relative errors in the neighborhood of 0.42%. Note that the symmetry is now around the origin.

Compared with Equation (2), it is much easier to see why Equation (3) would be a good approximation to $\tan(\pi x)$. Not only is the new approximation odd with the appropriate vertical asymptotes at $x = \pm\frac{1}{2}$, but the first, third, and fifth terms of the Maclaurin series are accurate to within 0.42%, 3.7%, and 6.8%, respectively. In particular, the derivative of the approximation in (3) evaluates to $2 + \frac{2}{3}\sqrt{3} \approx 3.1547$ at $x = 0$, a not wholly unreasonable approximation to π.

If we use a common alternative draftsman's construction with C at $(0, -1.75)$ instead of $(0, -\sqrt{3})$ and make the same shift, we get a tangent approximation of

$$\tan(\pi x) \approx \frac{2x(49+4\sqrt{49-132x^2})}{7(-16x^2+\sqrt{49-132x^2})},$$

which also has vertical asymptotes at $x = \pm\frac{1}{2}$. Furthermore, its first derivative evaluated at $x = 0$ is $\frac{22}{7} \approx 3.1429$, the most well-known rational approximation of π. Just as $\frac{22}{7}$ is accurate to 0.04%, this approximation is accurate to within about 0.04% near $x = 0$, though it is less accurate than Equation (3) farther out.

Back to Geometry. We now use the geometric interpretations of the algebraic transformations we undertook to obtain the improved tangent approximation from Equation (3) to improve the draftsman's construction.

We did three things to the tangent approximation in Equation (2) to obtain the approximation in Equation (3):

- replace x by $x + \frac{1}{2}$ in Equation (2),
- negate the fraction, and
- invert the negated fraction.

Let us see how these steps are reflected in a new construction. Recall that we obtained the continuous tangent approximation in Equation (2) via the substitution $x = \frac{2}{n}$, so when we replace x with $x + \frac{1}{2}$, we are replacing $\frac{2}{n}$ with $\frac{2}{n} + \frac{1}{2}$. In the algebraic analysis of the original construction, $\frac{2}{n}$ arose when we found the coordinates of point D_{n-2}, joining point C with the point with coordinates $(1 - 2(\frac{2}{n}), 0)$. If we replace $\frac{2}{n}$ with $\frac{2}{n} + \frac{1}{2}$ in this expression and simplify, we find that the coordinates $(1 - 2(\frac{2}{n}), 0)$ transform to the coordinates $(-\frac{4}{n}, 0)$.

More simply, the negation transforms these coordinates further to $(\frac{4}{n}, 0)$. Thus, rather than dividing the diameter into n pieces and counting two from the right, we will have to divide a *radius* into n pieces and count out four from the center. We will then join the original C with the point $(\frac{4}{n}, 0)$.

Finally, the inversion of the fraction corresponds to a reversal of the vertical and horizontal coordinates. We could incorporate a reflection around the line $y = x$, but it will be easier to measure the initial angle from the vertical rather than from the horizontal axis. This is where the extra step comes in, finding a point where the perpendicular bisector of the initial diameter meets the circle. If one thinks of the diameter as oriented horizontally, one must construct the point at the top of the circle.

Let us make these changes explicit in the improved draftsman's construction given below, and as pictured in Figure 6.

1. Find a diameter of the given circle, with endpoints A and B, and find its midpoint O.
2. Divide the radius OB into n equal segments $OR_1, R_1R_2, \ldots, R_{n-2}R_{n-1}, R_{n-1}B$.

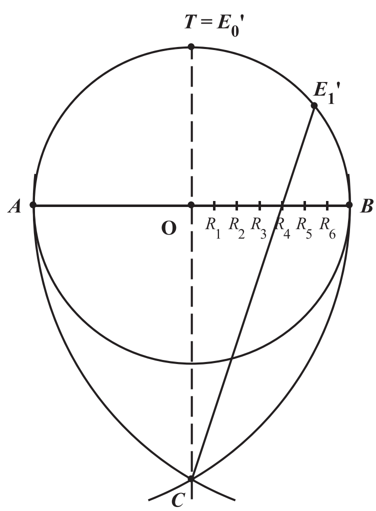

Figure 6 Steps 1–5 of the improved draftsman's construction with $n = 7$.

TABLE 2: Relative errors in constructing the first central angles using the improved draftsman's construction. For the $n = 360$, the construction does not produce an n-gon.

n	5	6	7	8	9	10
% Error in angle	-0.88	-0.79	-0.59	-0.42	-0.27	-0.16
n	15	20	25	30	60	*360*
% Error in angle	0.14	0.26	0.31	0.35	0.40	0.42

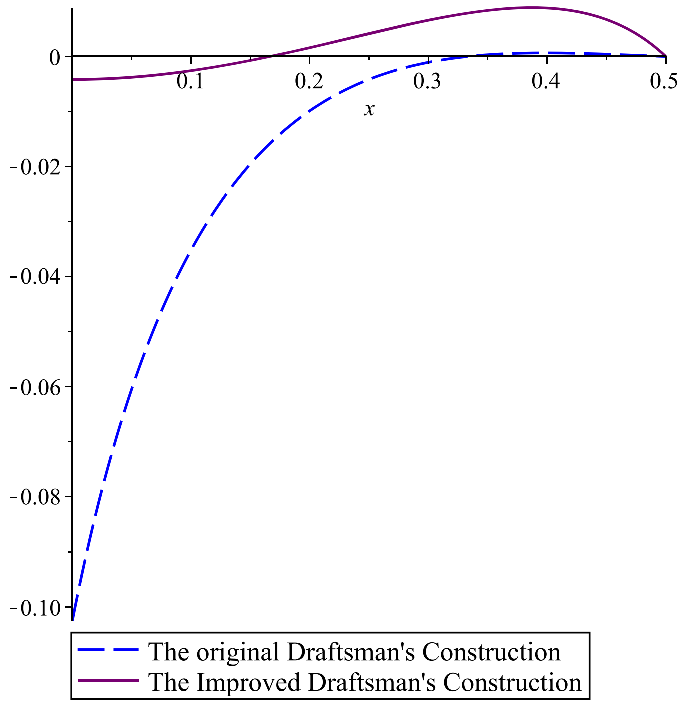

Figure 7 Relative errors in the size of the initial central angle using the two versions of the construction.

3. Use the compass to make two circular arcs with radius AB, one with center A, the other with center B. These arcs will meet at a point which we will call C directly below O.
4. Connect C to O and extend this to a point T on the circle on the side opposite AB from C.
5. Let CR_4 intersect the circle at point E'_1 on the side opposite AB from C. The segment TE'_1 will be the first side of the n-gon.
6. Having found two points of the n-gon, we want to continue around the circle, using the distance between $T = E'_0$ and E'_1 as a guide. Successively construct points $E'_2, E'_3, \ldots, E'_{n-1}$ by constructing circles centered at E'_i with radius $E'_{i-1}E'_i$. These smaller circles will intersect the large circle at points E'_{i-1}, E'_i, and E'_{i+1}.
7. The finished n-gon is $TE'_1E'_2 \cdots E'_{n-1}$. Note that points A and B are typically not part of the n-gon.

This revision actually makes things a little worse for $n = 5$, 6, and 7, and the two perform equally at $n = 8$, with the original construction overestimating and the revised version underestimating the central angle by precisely the same amount, about 0.42%.

Of course in practice any draftsman would inscribe a square and bisect angles precisely rather than use either version of this construction to inscribe an octagon.

However, for $n = 9$ and larger, the new version outperforms the original handily, as you can see from the graph in Figure 7 of percentage errors in constructing the first central angle. (Recall that $x = \frac{2}{n}$, so $n = 20$ would correspond to $x = 0.1$, for example.) While the original construction was exact at $n = 6$, the improved version is now exact at $n = 12$ and has less than 0.2% error for n between 10 and 17. It never exceeds 0.42% error in the initial angle for $n \geq 8$.

While the approximations using the original draftsman's construction got so bad that they weren't even producing n-gons any more by $n = 25$, that threshold is not reached until $n = 241$ with the improved draftsman's construction, well beyond what anyone would ever do by hand. In constructing a 25-gon with the improved draftsman's construction 24 of the angles are about 0.3% too large and the last one is about 7.5% too small.

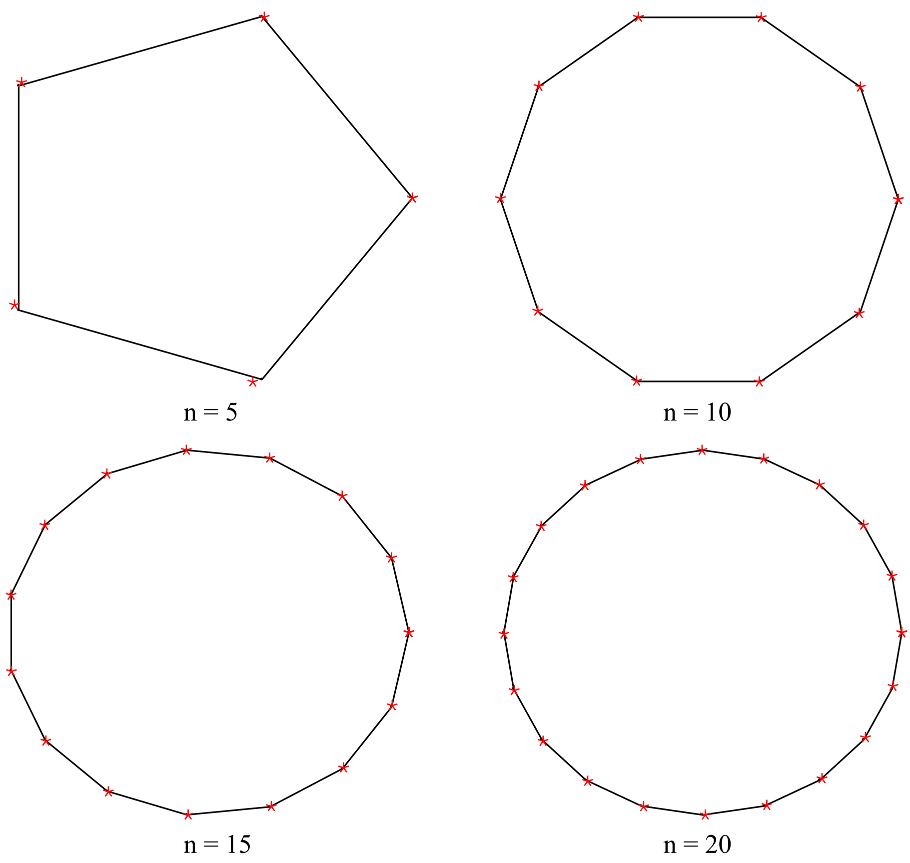

Figure 8 A regular n-gon is given in black, and the points generated by the improved draftsman's construction (rotated to share an orientation with Figure 3) are marked with red asterisks. Notice that the pentagon is slightly worse than the original version of the construction, and that for $n = 10$, 15, and 20, the improved construction is nearly perfect.

Thus, one would want to use the original draftsman's construction to inscribe pentagons and heptagons in circles (or, for pentagons, an exact construction); to use a common-sense exact construction for hexagons and octagons; and for anything with nine or more sides, to take the extra few steps to use the improved draftsman's construction.

A very brief history of the construction

With the introduction of the adjustable set square in the 1940s and—more importantly—computer-assisted design in the last part of the 20^{th} century, compass-and-straightedge constructions such as this have become unnecessary to modern engineers. This construction appears to be entirely absent from U.S. drafting textbooks of the past 70 years.

However, the draftsman's construction appears in dozens of engineering textbooks from the 1940s and earlier, going back to at least 1811 ([**2**]). Tracing its history further back than that is nearly impossible, since until the 19^{th} century, engineers learned elementary drafting mostly in master-apprentice relationships rather than from written treatises.

Despite its prevalence in engineering textbooks, the draftsman's construction has made very few appearances in the mathematical literature. I found mention in a single geometry textbook ([**9**]), which led me to a single article by Peter Merrotsy ([**7**]), who had learned of the construction from a friend who had trained as an engineer. If anyone reading this article knows more about the history of the draftsman's construction, I would be delighted to hear from you!

Acknowledgment Thanks to Lila Greco, the Kenyon student whose wonderful talk on Celtic knots introduced me to this topic, to my correspondents Reviel Netz, Bruce Shawyer, and Peter Merrotsy, and to my Kenyon colleagues, especially Judy Holdener, for being sounding boards as I was working on this project. Thanks particularly to Shawn Farnell, Tom Giblin, Josh Holden, Dan Laskin, John Noonan, Marie Snipes, Carolyn Yackel, and an exceptionally patient and thorough anonymous referee for careful readings of drafts of this paper.

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ACROSS

1. Past MAA president Deborah T. with an eponymous award for teaching
6. Letter between chi and omega
9. Harvard, Yale, Cornell, etc.
14. Senator Hatch, to friends
15. "I caught you!"
16. Make a processing error
17. First woman MAA president Dorothy L.
19. 𝄞
20. 0.1
21. Like a fingerprint found by dusting
22. Actors' organization acronym
23. Pool stick
24. Prefix for "morphic" or "geneous"
25. ___ *in Cleveland*
26. The "V" in VA
30. *A Christmas Story*'s Red Ryder ___
33. Essential ___ , like sandalwood and lavender
34. Fashion a Klein bottle out of yarn, say
35. Category theory co-founder and past MAA president
38. Exam for future docs
39. Plants whose leaves and stems are used in miso soup
40. Fables writer
41. Michael Jackson's scary opus
43. *George of the Jungle* character
44. ___ -do-well
45. Hex, Dec, Bin, ___
46. Old Dungeons & Dragons co.
49. Rich man, poor man, ___ man, thief
52. Overblown ruckus
54. *Night Music* playwright Clifford
55. Juggler and past MAA president
56. Tippe ___ and Tyler too
57. "Xanadu" by O.N.-J. & ___
58. Marry secretly
59. Add to the Constitution, say
60. ___ product
61. Past MAA president and co-editor of this MAGAZINE Lynn Arthur

DOWN

1. Boxcar riders of old
2. Indoor football league venue
3. Like some factory seconds: Abbr.
4. In perfect condition
5. "Who's on first, What's ___ "
6. Baudelaire, par exemple
7. ___ Tzu (toy dog breed)
8. Non-neutral particle
9. "Yes, count me in!"
10. Art gallery problem proposer and past MAA president
11. ___ of Man
12. *I Dream of Jeannie* actress Barbara
13. ___ shell crab
18. Fastener named for its shape
21. Plenty of, casually
24. Sea captains' posts
25. Past MAA president Edward V., whose apportionment method is used in the US
26. Bill of a cap
27. Santa ___ (California winds)
28. El ___ (climate change in the Pacific Ocean)
29. Type of function, e.g., greatest integer
30. Real estate ad abbr.
31. Hofstadter's *Gödel, Escher*, ___
32. ___ gum: thickening agent
33. Number of elements in a set
36. Königsberg bridge problem solver
37. Takes, in chess
42. Not owned, as a car
43. "I'm just ___ in the wheel"
45. To be, ___ to be
46. Largest non-Great Lake in the US, by volume
47. Circle, triangle, or square, to a kindergartner
48. Dorm room noodle
49. ___ Raton conference in discrete math
50. Cheese named for a town in the Netherlands
51. Kelly of *Singin' in the Rain*
52. ___ tie (official neckwear of Arizona)
53. Stop, in Turing's problem
55. Primary color

Math. Mag. **88** (2015) 270–271. doi:10.4169/math.mag.88.4.270.

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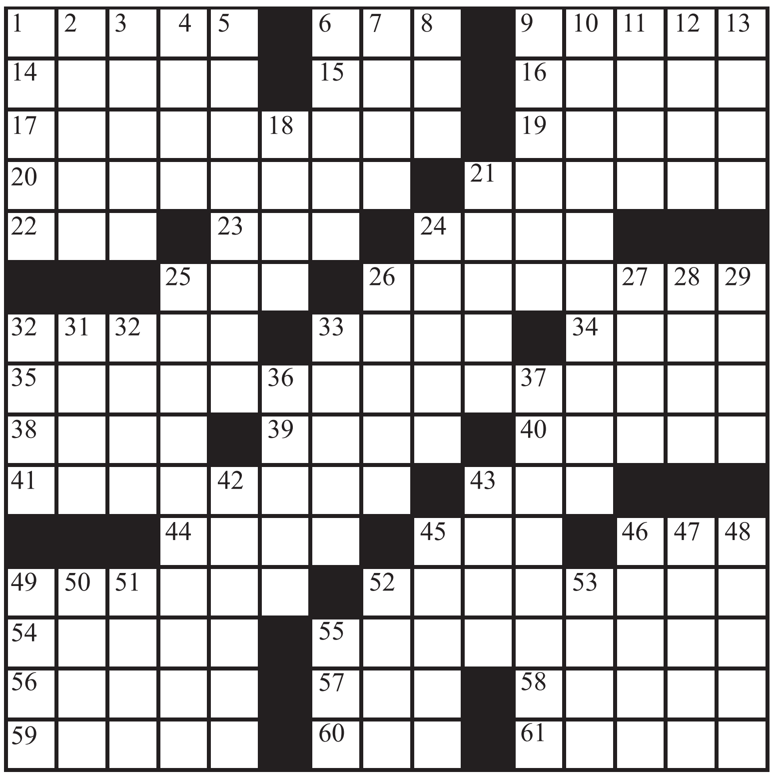

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Möbius Maps and Periodic Continued Fractions

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Most people meet continued fractions for the first time in number theory, but for well over 100 years, people have studied complex continued fractions of the form

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ddots}}, \tag{1}$$

where the a_i and b_j are complex numbers (possibly zero). The arguments used in number theory are inadequate for discussing these continued fractions, and a broader view is necessary. One such view is based on the use of Möbius maps, that is, on complex functions of the form

$$g(z) = \frac{az+b}{cz+d}, \tag{2}$$

where a, b, c and d are complex numbers with $ad - bc \neq 0$. To see why Möbius maps are relevant here, let $s_1(z) = (az+1)/z = a + 1/z$ and $s_2(z) = (bz+1)/z = b + 1/z$; then

$$s(z) = s_1\big(s_2(z)\big) = a + \cfrac{1}{b + \cfrac{1}{z}}$$

is a finite continued fraction and also a Möbius map, namely

$$s(z) = \frac{(ab+1)z + a}{bz+1}.$$

We will revisit Möbius maps after introducing some terminology about continued fractions that will allow us to state a theorem by Galois.

This paper is primarily about using Möbius transformations to analyze continued fractions of the form

$$[b_0, b_1, b_2, \ldots] = b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \ddots}} = \lim_{n\to\infty} [b_0, b_1, \ldots, b_n], \tag{3}$$

where the b_j are integers, with $b_1, b_2, \ldots$ positive and where

$$[b_0, b_1, \ldots, b_n] = b_0 + \cfrac{1}{b_1 + \cfrac{1}{\cdots + \cfrac{1}{b_n}}}. \tag{4}$$

Math. Mag. **88** (2015) 272–277. doi:10.4169/math.mag.88.4.272.
MSC: Primary 40A15, Secondary 30B70; 11A55.

The limit in (3) always exists, and $[b_0, b_1, \ldots, b_n]$ in (4) is calculated according to the usual rules of arithmetic.

A continued fraction $[b_0, b_1, b_2, \ldots]$ is *periodic* with period k if $b_n = b_{n+k}$ for all n and *eventually periodic* if $b_n = b_{n+k}$ for all sufficiently large n. We prefer this terminology (borrowed from dynamical systems) to the classical terminology for continued fractions which would be *purely periodic* and *periodic*, respectively. If $[b_0, b_1, b_2, \ldots]$ is periodic with period k, then we write $[\overline{b_0, \ldots, b_{k-1}}]$ for $[b_0, b_1, b_2, \ldots]$. Although in general b_0 can be any integer, if $[b_0, b_1, b_2, \ldots]$ is periodic with period k, then $b_0 = b_k \geqslant 1$. In particular, the value of a periodic continued fraction exceeds 1.

A real number x is a *quadratic irrational* if and only if it is irrational and also the zero of a quadratic polynomial P with integer coefficients. The polynomial P here is unique up to a scalar multiple, and the second zero of P is the *algebraic conjugate* x^* of x. Equivalently, a quadratic irrational is an irrational number that can be written as either $q + \sqrt{r}$ or $q - \sqrt{r}$, where q and r are rational; then $q + \sqrt{r}$ and $q - \sqrt{r}$ are the *algebraic conjugates* of each other.

It is well known that each irrational number has a unique infinite continued fraction expansion of the form (3). Suppose now that x is irrational and that $x = [b_0, b_1, b_2, \ldots]$. The significance of quadratic irrationals is that $[b_0, b_1, b_2, \ldots]$ *is eventually periodic if and only if* x *is a quadratic irrational* and that $[b_0, b_1, b_2, \ldots]$ *is periodic if and only if* x *is a quadratic irrational whose algebraic conjugate* x^* *lies in* $(-1, 0)$. Euler proved that a periodic continued fraction is a quadratic irrational, and the converse was proved by Lagrange; for proofs of this see, for example, [**3**, Theorems 176 and 177], [**7**, p. 119], and [**6**, pp. 40–41]. The last result here is due to Galois.

Galois studied periodic continued fractions and proved the following result (see, for example, [**6**, p. 46]).

Galois' theorem. If $x = [\overline{b_0, \ldots, b_{k-1}}]$ then $[\overline{b_{k-1}, \ldots, b_0}] = -1/x^*$.

Note that if the continued fraction expansion of x is periodic, then $x > 1$. Similarly, $-1/x^* > 1$ so that $x^* \in (-1, 0)$.

The purpose of this paper is to highlight the application of Möbius transformations to continued fractions by using them to prove Galois' theorem (which is usually proved by the manipulation of solutions of recurrence relations). The use of Möbius transformations gives additional insight by showing the significance of the map $z \mapsto -1/z$ and also provides a proof when the b_i are *real* numbers greater than 1 (and not restricted to integers). We end this section with a simple example in which these ideas are completely transparent and which are used to prove Galois' theorem in a special case.

Example. Let a and b be positive integers, and let $\alpha = [\overline{a, b}]$. Simple substitution yields

$$\alpha = a + 1/(b + 1/[\overline{a, b}]) = a + 1/(b + 1/\alpha)$$

so that α is a fixed point of s, where $s(z) = a + 1/(b + 1/z)$. Now the fixed points of s are the solutions of $bz^2 - abz - a = 0$, so the roots of this equation are α and its algebraic conjugate α^*. Since $\alpha\alpha^* = -a/b < 0$, we see that $\alpha > 0 > \alpha^*$. Now let $\beta = [\overline{b, a}]$. Then, similarly, β is a fixed point of the map $z \mapsto b + 1/(a + 1/z)$, and β and β^* are the roots of the equation $az^2 - abz - b = 0$. Note that $\beta > 0 > \beta^*$. By considering the transformation $w = -1/z$, we find that $\{\alpha, \alpha^*\} = \{-1/\beta, -1/\beta^*\}$ so that $\beta = -1/\alpha^*$, and this is Galois' result in this case.

The broader view

We begin by recalling the theory of Möbius maps. Let $\mathbb{C}$ be the complex plane. We adjoin a "new" point, which we label ∞, to $\mathbb{C}$ to form the *extended complex plane* $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$. A map g with domain $\mathbb{C}_\infty$ is a *Möbius map* if it can be written in the form (2) where a, b, c, and d are complex numbers with $ad - bc \neq 0$. If $c \neq 0$, we interpret (2) to say that $g(\infty) = a/c$ and $g(-d/c) = \infty$. If $c = 0$, then $g(\infty) = \infty$. Then each Möbius map g is a bijection of $\mathbb{C}_\infty$ onto itself, and the inverse of a Möbius map is a Möbius map. Moreover, the set of Möbius maps is a group under composition.

The Möbius map g in (2) preserves the extended real axis $\mathbb{R} \cup \{\infty\}$, which we denote by $\mathbb{R}_\infty$, if and only if a, b, c, and d are real numbers. Further, as

$$\mathrm{Im}[g(z)] = \frac{(ad - bc)\mathrm{Im}[z]}{|cz + d|^2},$$

we see that g preserves the upper half-plane $\mathbb{H} = \{x + iy : y > 0\}$ if and only if $ad - bc > 0$. Note that if

$$h(z) = \frac{az + b}{cz + d},$$

where a, b, c, and d are real and $ad - bc = -1$, then h can be written as

$$h(z) = \frac{iaz + ib}{icz + id}, \quad (ia)(id) - (ib)(ic) = 1,$$

and h interchanges $\mathbb{H}$ with the lower half-plane $\mathbb{H}^-$ (defined by $y < 0$). This is the case, for example, when $h(z) = 1/z$. On the other hand, $z \mapsto -1/z$ preserves both $\mathbb{H}$ and $\mathbb{H}^-$.

The remarks on metric spaces and hyperbolic geometry that follow are not necessary for the proofs in the paper, but they do explain the main ideas and motivation that lie behind this work. It is easy to make $\mathbb{C}_\infty$ into a metric space, for we can project it (using stereographic projection) onto the unit sphere $\mathbb{S}$ in $\mathbb{R}^3$ and then transfer the Euclidean metric from $\mathbb{S}$ to a metric χ on $\mathbb{C}_\infty$. Now that $(\mathbb{C}_\infty, \chi)$ is a metric space, we can show that each Möbius map g is a homeomorphism of $\mathbb{C}_\infty$ onto itself. Better still, with an appropriate definition (which we omit here), the group of Mobius maps (under composition) is the group of all conformal mappings of $(\mathbb{C}_\infty, \chi)$ onto itself. With this, we have set up the equipment to deal with continued fractions of the form (1).

Observe that the boundary of $\mathbb{H}$ is $\mathbb{R}_\infty$ and that $\mathbb{H}$, with the metric $|dz|/y$, is one of the standard models of the hyperbolic plane. Moreover, the (conformal) isometries of the hyperbolic plane are precisely the Möbius maps that leave $\mathbb{H}$ invariant. Indeed, the hyperbolic metric on $\mathbb{H}$ is a "natural" metric from the point of view of complex analysis since it is (up to a scalar multiple) the only metric for which the conformal self-maps of $\mathbb{H}$ are isometries. Although we shall not use hyperbolic geometry in this article, we would be remiss if we did not mention the fact that Möbius maps (when suitably defined) exist in Euclidean spaces of all dimensions and form the conformal isometry groups of hyperbolic space of the appropriate dimension. Sadly, the usual introduction of Möbius maps acting on $\mathbb{C}_\infty$ (where they are not isometries) completely misses this point.

The modular group

The *modular group* Γ is the group of Möbius maps of the form (2) where now a, b, c, and d are *integers* and $ad - bc = 1$. For example, the map $s(z) = a + 1/(b + 1/z)$ in the example is in the modular group. The modular group acts on the hyperbolic plane $\mathbb{H}$ as a group of isometries of the hyperbolic metric, and its action on the boundary $\mathbb{R} \cup \{\infty\}$ of $\mathbb{H}$ is intimately connected to the theory of continued fractions. Indeed, a modern view of continued fractions is to see them from the geometric perspective of the action of the modular group Γ on the boundary of $\mathbb{H}$. To study continued fractions by using only real methods is, in effect, ignoring the fact that the real axis is the boundary of $\mathbb{H}$ and that the essential mathematics here is to be found in the action of Γ on $\mathbb{H}$. To see how these ideas are used to study continued fractions see, for example, [**1, 2, 5, 8**].

Of particular interest to us are the *loxodromic isometries* of $\mathbb{H}$. These are the Möbius maps that preserve $\mathbb{H}$, and have two distinct real fixed points (for example, $z \mapsto 2z$ that fixes 0 and ∞). With these, we can see quadratic irrationals in a different light for it is a (nontrivial) fact that *a real x is a quadratic irrational if and only if it is a fixed point of some loxodromic element g of the modular group* Γ, and then *its algebraic conjugate x^* is the second fixed point of g*. This fact makes it completely transparent why, for example, quadratic irrationals have a role to play in continued fraction theory whereas cubic (and higher order) irrationals do not. Indeed, in [**4**], Khinchin proves that the real number x has an eventually periodic continued fraction expansion if and only if it is a quadratic irrational and then writes (on page 50) that "no proofs analogous to this are known for continued fractions representing algebraic irrational numbers of higher degrees." The reason for this is now apparent for the fundamental result here is that a number is a quadratic irrational if and only if it is fixed by a loxodromic element in the modular group.

There is one more fact that is crucial to our argument: If g is a loxodromic map with fixed points u and v, then one of u and v, let us say u, is an *attracting fixed point*, and then v is a *repelling fixed point*. This means that if g^n is the n-iterate of a loxodromic map g (that is the composition obtained by applying g exactly n times), then $g^n(z) \to u$ if $z \neq v$. For example, ∞ and 0 are the attracting and repelling, respectively, fixed points of $z \mapsto \lambda z$ when $|\lambda| > 1$. Informally, a fixed point w of a general map f is attracting or repelling according as $|f'(w)| < 1$ or $|f'(w)| > 1$.

Finally, the *extended modular group* Γ^* (which we shall need later) is the group of all maps $z \mapsto (az + b)/(cz + d)$, where a, b, c, and d are integers but now $|ad - bc| = 1$. It is easy to see that Γ^* is a group, with Γ a subgroup of index two in Γ^*. Note that if a is an integer, then the map $z \mapsto a + 1/z$ is in Γ^*.

The basic lemma

Our proof of Galois' theorem is based on the following simple lemma that, since the b_j need not be integers, generalizes the number-theoretic notion of an algebraic conjugate in geometric terms. In this lemma, $s_1 s_2(z) = s_1\big(s_2(z)\big)$, and so on.

Lemma. A finite composition, say $S = s_1 \cdots s_k$, of k maps of the form $s : z \mapsto b + 1/z$, where $b \geqslant 1$, has an attracting fixed point, say ζ, in $(1, +\infty)$, and a repelling fixed point, say $\tilde{\zeta}$, in $(-1, 0)$.

Proof. Let $S = s_1 \cdots s_k$, where $s_j(z) = b_j + 1/z$ and $b_j \geqslant 1$. As each s_j maps $[1, +\infty)$ onto a bounded subinterval of $(1, +\infty)$, the composition S does the same, so we conclude (from the usual elementary fixed point theorem) that S has a fixed point, say ζ, in $(1, +\infty)$. Since $|s_j'(x)| < 1$ on $(1, +\infty)$, the chain rule implies that $|S'(\zeta)| < 1$ so that ζ is an attracting fixed point of S. Now let $\tilde{S} = s_k \cdots s_1$. Then, for exactly the same reason, $\tilde{S}$ has an attracting fixed point, say $\tilde{\zeta}$, in $(1, +\infty)$. Let

$\sigma(z) = -1/z$. Then $\sigma s_j = s_j^{-1}\sigma$ so that $\sigma S = \tilde{S}^{-1}\sigma$. It follows easily that S also fixes $\sigma(\tilde{\zeta})$ and that this is the repelling fixed point of S in $(-1, 0)$.

If the b_j in the lemma are positive integers, then S is in the extended modular group, so the quadratic equation for the fixed points of S has integer coefficients and then ζ and $\tilde{\zeta}$ are algebraic conjugates in the usual sense. As we noted in our discussion of attracting and repelling fixed points, we see that $S^n(z) \to \zeta$ as $n \to \infty$, providing that $z \neq \tilde{\zeta}$. Finally, we recall that we can compute the composition of Möbius maps by multiplying their associated matrices. Thus, the matrix for S is the product of the matrices for the s_j, namely

$$\begin{pmatrix} b_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} b_k & 1 \\ 1 & 0 \end{pmatrix}.$$

We do not have to compute this matrix, but we do note that it has determinant $(-1)^k$. We conclude that S is in Γ if k is even and in Γ^* (but not in Γ) if k is odd.

The proof of Galois' theorem

Now consider the collection of maps $s_j(z) = b_j + 1/z$, $j = 0, 1, 2, \ldots$, where, for each j, $b_j \geqslant 1$ (note that the b_j may, but need not, be positive integers). Then, by definition,

$$[b_0, b_1, b_2, \ldots] = \lim_{n\to\infty} s_0 \cdots s_n(\infty).$$

Suppose now that $b_0, b_1, b_2, \ldots$ is periodic with period k. Let $S = s_0 \cdots s_{k-1}$, and let ζ and $\tilde{\zeta}$ be the attracting and repelling fixed points of S. Thus, $\zeta > 1$ and $\tilde{\zeta} \in (-1, 0)$. Next, let

$$K = \{\infty, s_0(\infty), s_0 s_1(\infty), \ldots, s_0 s_1 \cdots s_{k-2}(\infty)\}.$$

Then, by the lemma, $K \subset [1, +\infty)$, and since $\tilde{\zeta} < 0$, we see that $S^n(z) \to \zeta$ for each z in K. This is the same as saying that $s_0 \cdots s_n(\infty) \to \zeta$ as $n \to \infty$; thus,

$$[\overline{b_0, \ldots, b_{k-1}}] = [b_0, b_1, \ldots] = \zeta.$$

The argument in the proof of the lemma, combined with the discussion here, shows that

$$[\overline{b_{k-1}, \ldots, b_0}] = \tilde{\zeta}.$$

Since S fixes $-1/\tilde{\zeta}$ (see the proof of the lemma) we see that $-1/\tilde{\zeta} = \zeta^*$ (the algebraic conjugate of ζ), and hence,

$$[\overline{b_{k-1}, \ldots, b_0}] = -1/\zeta^*,$$

which is Galois' result. In fact, this proves more since our argument does not require the b_j to be integers.

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Summary. We describe some of the relationships that exist between quadratic irrationals, continued fractions, Möbius maps, and hyperbolic geometry, and we illustrate these by giving a simple geometric proof of Galois' result on dual continued fractions.

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Proof Without Words: Sums of Odd and Even Squares

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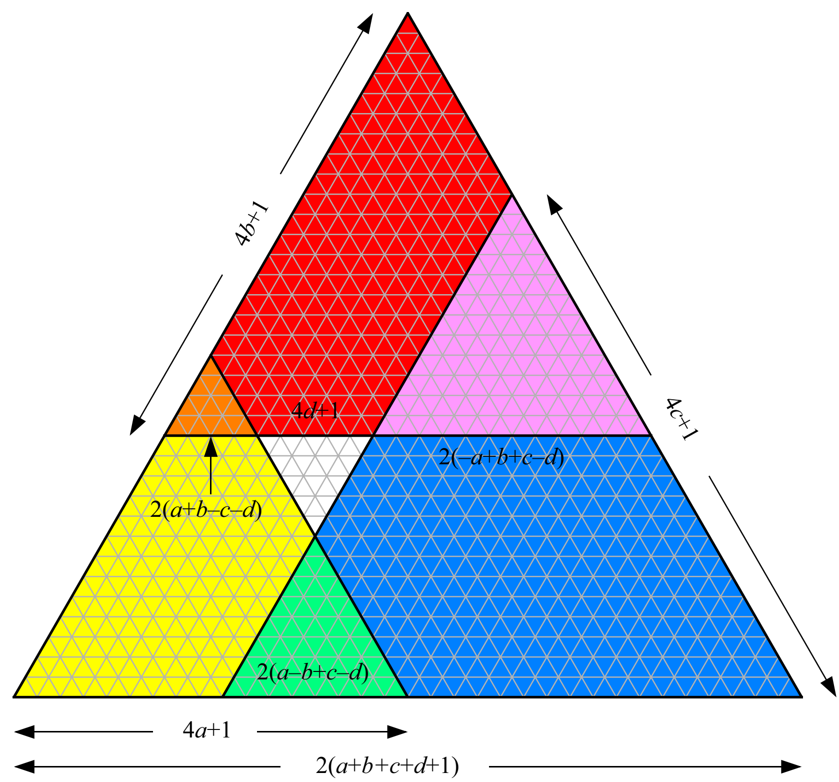
$$(4a+1)^2+(4b+1)^2+(4c+1)^2+(4d+1)^2$$
$$=[2(a+b+c+d+1)]^2+[2(a+b-c-d)]^2$$
$$+[2(a-b+c-d)]^2+[2(-a+b+c-d)]^2.$$

Proof (by inclusion–exclusion, where each Δ or ∇ is counted as 1), e.g., for $(a,b,c,d) = (4,5,6,1)$:



$$[2(a+b+c+d+1)]^2=(4a+1)^2+(4b+1)^2+(4c+1)^2+(4d+1)^2$$
$$-[2(a+b-c-d)]^2-[2(a-b+c-d)]^2$$
$$-[2(-a+b+c-d)]^2.$$

Math. Mag. **88** (2015) 278–279. doi:10.4169/math.mag.88.4.278.
MSC: Primary 05A19; Secondary 00A05.

Notes:

1. The identity actually holds for all real numbers a, b, c, d (expand and simplify both sides), with figures different from the one above needed to illustrate many choices of a, b, c, d.
2. It is easy to show that the terms on the left side of the identity are distinct if and only if the terms on the right side are distinct.
3. Consequently, if a, b, c, d are integers (positive, negative, or zero), then we have the following theorem of M. Hirschhorn [**1**, **2**]:

Theorem. *Every sum of four distinct odd squares is the sum of four distinct even squares.*

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ROGER B. NELSEN (MR Author ID: 237909) is a professor emeritus at Lewis & Clark College, where he taught mathematics and statistics for 40 years.

1 H	2 A	3 I	4 M	5 O		6 P	7 S	8 I		9 I	10 V	11 I	12 E	13 S
14 O	R	R	I	N		15 O	H	O		16 M	I	S	D	O
17 B	E	R	N	S	18 T	E	I	N		19 G	C	L	E	F
20 O	N	E	T	E	N	T	H		21 L	A	T	E	N	T
22 S	A	G		23 C	U	E		24 H	O	M	O			
			25 H	O	T		26 V	E	T	E	R	27 A	28 N	29 S
30 B	31 B	32 G	U	N		33 O	I	L	S		34 K	N	I	T
35 S	A	U	N	D	36 E	R	S	M	A	37 C	L	A	N	E
38 M	C	A	T		39 U	D	O	S		40 A	E	S	O	P
41 T	H	R	I	42 L	L	E	R		43 A	P	E			
			44 N	E	E	R		45 O	C	T		46 T	47 S	48 R
49 B	50 E	51 G	G	A	R		52 B	R	O	U	53 H	A	H	A
54 O	D	E	T	S		55 R	O	N	G	R	A	H	A	M
56 C	A	N	O	E		57 E	L	O		58 E	L	O	P	E
59 A	M	E	N	D		60 D	O	T		61 S	T	E	E	N

A Combinatorial Formula for Powers of 2×2 Matrices

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In this MAGAZINE, Williams [**6**] derived a formula for the nth power of a 2×2 matrix based on the eigenvalues of the matrix. Let $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with eigenvalues α and β, then the formula is

$$A^n = \begin{cases} \alpha^n \left(\frac{A-\beta I}{\alpha-\beta}\right) + \beta^n \left(\frac{A-\alpha I}{\beta-\alpha}\right), & \text{if } \alpha \neq \beta \\ \alpha^{n-1}\left(nA-(n-1)\alpha I\right), & \text{if } \alpha = \beta. \end{cases}$$

Subsequently, McLaughlin [**4**] used induction to derive a different formula for the nth power of the matrix A that depended on its determinant and trace.

In this note, using properties of binomial coefficients we derive a purely combinatorial formula for the nth power of A. The entries of the power matrix are represented as sums of products that depend only on the entries of the original matrix A. In addition, we apply the formula, stated in the following theorem, to prove two combinatorial identities, one of which is related to the Fibonacci numbers.

Theorem 1. *Assume* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *and let* $A^n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$, *where n is a positive integer. Then*

$$a_n = \sum_{k=0}^{n-1}\sum_{j=0}^{k} \binom{n-k}{j}\binom{k-1}{k-j} a^{n-k-j} b^j c^j d^{k-j},$$

$$b_n = \sum_{k=0}^{n-1}\sum_{j=0}^{k} \binom{n-1-k}{j}\binom{k}{k-j} a^{n-1-k-j} b^{j+1} c^j d^{k-j},$$

$$c_n = \sum_{k=0}^{n-1}\sum_{j=0}^{k} \binom{n-1-k}{j}\binom{k}{k-j} d^{n-1-k-j} c^{j+1} b^j a^{k-j}, \text{ and}$$

$$d_n = \sum_{k=0}^{n-1}\sum_{j=0}^{k} \binom{n-k}{j}\binom{k-1}{k-j} d^{n-k-j} c^j b^j a^{k-j}.$$

Observe the combinatorial formulas for a_n and d_n are identical except the order of the entries a, b, c, and d is reversed to read d, c, b, and a. The same observation applies to b_n and c_n This unexpected symmetry property is called *algebraic centrosymmetry* and "formally" holds for all powers of all square matrices! See [**3**].

Math. Mag. **88** (2015) 280–284. doi:10.4169/math.mag.88.4.280.
MSC: Primary 05A05.

Proof. The formula will be derived by applying properties of binomial coefficients and the definition of matrix multiplication. First, we will index the entries of the matrix using a binary scheme. That is, the entries of matrix A will be denoted by a_{ij} where $0 \leqslant i, j \leqslant 1$ For example,

$$A = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix}, \text{ and thus, } A^2 = \begin{pmatrix} a_{00}^2 + a_{01}a_{10} & a_{00}a_{01} + a_{01}a_{11} \\ a_{11}a_{10} + a_{10}a_{00} & a_{11}^2 + a_{10}a_{01} \end{pmatrix}.$$

Note by the definition of matrix multiplication each entry A^2_{ij} of A^2 is of the form $a_{ik}a_{kj}$, where k is 0 or 1. We can encode this term by the sequence (i, k, j). In general, in the nth power of matrix A the entry A^n_{ij} is a sum of 2^{n-1} terms of the form $a_{ik_1}a_{k_1k_2}a_{k_2k_3}\cdots a_{k_{n-1}j}$, where $0 \leqslant k_i \leqslant 1$ for $i = 1$ to $n-1$. Thus, we can encode each A^n_{ij} as the binary sequence of length $n+1$, $(i, k_1, k_2, k_3, \ldots, k_{n-1}, j)$.

Now we have reduced our problem to counting like terms and expressing each entry of A^n as a sum of products of powers of the entries in the original matrix A. In addition to satisfying the recurrence relation $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$, the following combinatorial properties of the binomial coefficients $\binom{n}{k}$ and combinations with repetition will be applied ([**1**], [**5**]).

a) The number of positive integer solutions $(x_1, x_2, x_3, \ldots, x_j)$ of the linear equation $x_1 + x_2 + x_3 + \cdots + x_j = k$, where j and k are positive integers, is $\binom{k-1}{j-1} = \binom{k-1}{k-j}$.

b) The number of ways of choosing k integers, no two consecutive, from the set $\{1, 2, 3, \ldots, n\}$ is $\binom{n-k+1}{k}$.

We will now derive the formula for the entries A^n_{00} and A^n_{01} in the first row of A^n, the other two entries will follow by symmetry.

Case 1: Terms of the form $S = (0, k_1, k_2, k_3, \ldots, k_{n-1}, 0)$, where k_i is 0 or 1 for $1 \leqslant i \leqslant n-1$.

The terms in the entry A^n_{00} are products of the original entries of matrix A, namely, a_{00}, a_{01}, a_{10}, and a_{11}. These products are encoded by the sequence of subscripts $S = (0, k_1, k_2, k_3, \ldots, k_{n-1}, 0)$. Each consecutive pair of subscripts represents one of the entries a_{ij} in A. For the sequence S, the product begins with $a_{0k_1}$and ends with $a_{k_{n-1}0}$. To determine how many times each entry a_{ij} occurs in the product, we must count how many times the subscripts *ij* occur as consecutive pairs in the sequence S. Thus, if $S = (0, 0, 0, \ldots 0)$, then we have $n+1$ zeros, making up n pairs of consecutive zeros corresponding to the leading term a^n_{00} There remain $2^{n-1} - 1$ sequences S to consider, and each of these must contain at least one 1. Let k be the number of 1's in such a sequence, and let j be the number of blocks of 1's (groups of consecutive 1's separated on both ends by at least one 0). Thus, k can be expressed as an ordered partition of j positive integers $(b_1, b_2, b_3, \ldots, b_j)$ representing the block sizes:

$$b_1 + b_2 + b_3 + \cdots + b_j = k.$$

The number of ways this can happen is $\binom{k-1}{j-1} = \binom{k-1}{k-j}$. For each of the corresponding sequences there will be a 0 immediately preceding and following every block of 1's. This means the subscripts 01 and 10 will each occur j times as consecutive pairs in the sequence S. Thus, the corresponding term will contain the product $a_{01}^{j}a_{10}^{j}$.

Next, we must determine the number of times the subscripts 11 and 00 occur in S. In each block of size b, there will be $b-1$ consecutive pairs of 1's. Summing over all the block sizes we obtain $k-j$. Thus, the corresponding term will contain the factor a_{11}^{k-j}. Since the total number of consecutive pairs in S is n, we can obtain the number of subscripts containing 00 as consecutive pairs by subtracting the three results already computed: $n-(j+j+k-j)=n-k-j$. Thus, the corresponding term will contain the factor a_{00}^{n-k-j}.

Finally, the blocks must be nonconsecutive and separated by a variable number of 0's. Ignoring the 0's on both ends of S, we consider the interior $n-1$ bits. Label the bits using the set $\{1,2,3,\ldots,n-1\}$. We want to count the number of ways of picking j nonconsecutive blocks. We can reduce this problem to finding the number of ways to select j numbers, no two consecutive from the set $\{1,2,3,\ldots,n-1-(k-j)\}$. This is done by reducing each block size to size 1, which is equivalent to subtracting $k-j$ from $n-1$. Thus, the number of ways to choose j numbers from this set, no two consecutive, is $\binom{n-1-(k-j)-j+1}{j}=\binom{n-k}{j}$.

Summing over all k and j and letting $a=a_{00}$, $b=a_{01}$, $c=a_{10}$, and $d=a_{11}$, we obtain

$$a_n=\sum_{k=0}^{n-1}\sum_{j=0}^{k}\binom{n-k}{j}\binom{k-1}{k-j}a^{n-k-j}b^{j}c^{j}d^{k-j}.$$

Case 2: Terms of the form $S=(0,k_1,k_2,k_3,\ldots,k_{n-1},1)$, where k_i is 0 or 1 for $1\leqslant i\leqslant n-1$.

The terms in the entry A_{01}^{n} are determined by a similar argument as in Case 1 for A_{00}^{n} except we must consider two cases regarding the last block of 1's. Since the sequence S ends in a 1, the last block of 1's is either (i) separated by at least one 0 from the last 1 of S, or (ii) no 0's lie between the last block of 1's and the last 1 of S. From Case 1 we know that the number of ways of picking j nonconsecutive blocks of 1's is $\binom{n-k}{j}$. This is equivalent to picking j numbers, no two consecutive, from the set $\{1,2,3,\ldots,n-1-(k-j)\}$. Either the last number, $n-1-(k-j)$, from the set was picked or not picked. If the last number was not picked, this means (i) at least one 0 separates the last block from the last 1 of S. Otherwise, (ii) the sequence S ends with at least two consecutive 1's. So, Case (i) means we are picking j objects (nonconsecutive) from one less member of our set, and Case (ii) means we are picking $j-1$ objects (nonconsecutive) from one less member of our set. Thus, there are $\binom{n-1-k}{j}$ ways to do this for Case (i), and $\binom{n-1-k}{j-1}$ ways for Case (ii).

Next we will sum the Cases (i) and (ii) to obtain the entry A_{01}^{n}.

Case (i) Summing over all k and j and letting $a=a_{00}$, $b=a_{01}$, $c=a_{10}$, and $d=a_{11}$, we obtain

$$\sum_{k=0}^{n-1}\sum_{j=0}^{k}\binom{n-1-k}{j}\binom{k-1}{k-j}a^{n-1-k-j}b^{j+1}c^{j}d^{k-j}.$$

This is the sum of terms in A_{01}^{n} whose sequence S ends with the consecutive pair 01. This means we gained an increase of 1 in the exponent of $b=a_{01}$ and a decrease of 1 in the exponent of $a=a_{00}$.

Case (ii) Summing over all k and j and letting $a = a_{00}$, $b = a_{01}$, $c = a_{10}$, and $d = a_{11}$, we obtain

$$\sum_{k=0}^{n-1}\sum_{j=0}^{k}\binom{n-1-k}{j}\binom{k-1}{k-1-j}a^{n-1-k-j}b^{j+1}c^{j}d^{k-j}.$$

This is the sum of terms in A^n_{01} whose sequence S ends with the consecutive pair 11. This means we gained an increase of 1 in the exponent of $d = a_{11}$ and a decrease of 1 in the exponent of $c = a_{10}$.

Adding Cases (i) and (ii) and applying the recurrence relation for the binomial coefficients, we obtain

$$b_n = \sum_{k=0}^{n-1}\sum_{j=0}^{k}\binom{n-1-k}{j}\binom{k}{k-j}a^{n-1-k-j}b^{j+1}c^{j}d^{k-j}.$$

Finally, entries A^n_{10} and A^n_{11} of A^n can be easily obtained by observing that if we interchange 0 and 1 in Case 1 and Case 2 the same formulas are derived except the entries of the original matrix a and d are interchanged, as well as the entries b and c. Thus, we obtain

$$c_n = \sum_{k=0}^{n-1}\sum_{j=0}^{k}\binom{n-1-k}{j}\binom{k}{k-j}d^{n-1-k-j}c^{j+1}b^{j}a^{k-j}$$

$$d_n = \sum_{k=0}^{n-1}\sum_{j=0}^{k}\binom{n-k}{j}\binom{k-1}{k-j}d^{n-k-j}c^{j}b^{j}a^{k-j}. \qquad \blacksquare$$

Applications to Combinatorial Identities

The Fibonacci sequence f_n is defined as follows: $f_0 = 0$, $f_1 = 1$, $f_n = f_{n-1} + f_{n-2} (n \geqslant 2)$.

The Fibonacci numbers are related to powers of 2×2 matrices by the following matrix equation that can be proved by induction ([**2**], [**4**]):

$$\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}^n = \begin{pmatrix}f_{n+1} & f_n\\ f_n & f_{n-1}\end{pmatrix}.$$

If we apply our theorem with $a = 1$, $b = 1$, $c = 1$, $d = 0$ and compute the entry, say b_n, then we obtain as a special case the following known combinatorial identity ([**2**], [**5**]) expressing the Fibonacci number f_n as a sum of binomial coefficients:

$$f_n = \sum_{k=0}^{n-1}\sum_{j=0}^{k}\binom{n-1-k}{j}\binom{k}{k-j}1^{n-1-k-j}1^{j+1}1^{j}0^{k-j}$$

$$= \sum_{k=0}^{n-1}\binom{n-1-k}{k} = \sum_{k=0}^{\lfloor (n-1)/2\rfloor}\binom{n-1-k}{k}.$$

As a second application we will use Williams' eigenvalue formula and apply it to our theorem to obtain another combinatorial identity. Consider the following matrix equation that follows immediately from Williams' eigenvalue formula (both eigenvalues in this example are equal to 1):

$$\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}^n = \begin{pmatrix} n+1 & n \\ -n & 1-n \end{pmatrix}.$$

If we apply our theorem with $a = 2, b = 1, c = -1, d = 0$ and compute the entry, say b_n, then we obtain the following combinatorial identity [**4**]:

$$\begin{aligned} n &= \sum_{k=0}^{n-1} \sum_{j=0}^{k} \binom{n-1-k}{j} \binom{k}{k-j} 2^{n-1-k-j} 1^{j+1} (-1)^j 0^{k-j} \\ &= \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k \binom{n-1-k}{k} 2^{n-1-2k}. \end{aligned}$$

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Summary. Formulas for the nth power of a 2×2 matrix are known and typically depend on either computing the determinant of the matrix or its eigenvalues. In this note we derive a direct combinatorial formula for the nth power of the matrix that does not depend on any auxiliary computation. Applications include the derivation of a combinatorial identity involving the Fibonacci numbers.

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PROBLEMS

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Proposals

To be considered for publication, solutions should be received by March 1, 2016.

1976. *Proposed by Ángel Plaza, Universidad de las Palmas de Gran Canaria, Las Palmas, Spain.*

Let $\{a_n\}_{n\geq 1}$ be the sequence of real numbers defined by $a_1 = 3$ and for $n \geq 2$, $a_{n+1} = \frac{1}{2}(a_n^2 + 1)$. Evaluate

$$\sum_{k=1}^{\infty} \frac{1}{1+a_k}.$$

1977. *Proposed by Marcel Chirita, Bucharest, Romania.*

Let $ABCD$ be a tetrahedron inscribed in a sphere $\mathcal{S}$ of radius R. For every point M in space, define $f(M) = AM^2 + BM^2 + CM^2 + DM^2$. Suppose that $f(M)$ is constant for all points in $\mathcal{S}$.

(a) Calculate $f(M)$.

(b) Prove that $AB = CD$, $AC = BD$, and $AD = BC$; that is, prove that $ABCD$ is an isosceles tetrahedron.

Math. Mag. **88** (2015) 285–293. doi:10.4169/math.mag.88.4.285.

We invite readers to submit problems believed to be new and appealing to students and teachers of advanced undergraduate mathematics. Proposals must, in general, be accompanied by solutions and by any bibliographical information that will assist the editors and referees. A problem submitted as a Quickie should have an unexpected, succinct solution. Submitted problems should not be under consideration for publication elsewhere.

Solutions should be written in a style appropriate for this MAGAZINE.

New proposals should be mailed electronically to Eugen Ionascu at math@ejionascu.ro. *Solutions should be mailed to Bernardo M. Ábrego, Problems Editor, Department of Mathematics, California State University, Northridge, 18111 Nordhoff St., Northridge, CA 91330-8313, or mailed electronically (ideally as a LaTeX or pdf file) to* mathmagproblems@csun.edu. *All communications, written or electronic, should include* **on each page** *the reader's name, full address, and an e-mail address and/or FAX number.*

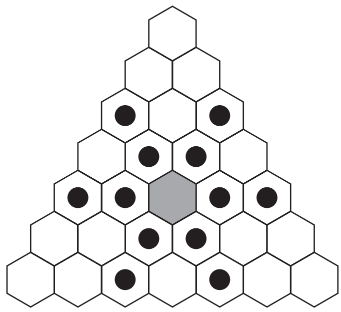
1978. *Proposed by George Apostolopoulos, Messolonghi, Greece.*

Evaluate

$$\lim_{n\to\infty}\left[\int_1^{e^2}\left(\frac{\ln x}{x}\right)^n dx\right]^{1/n}.$$

1979. *Proposed by Dan Ullman, George Washington University, Washington, DC, and Stan Wagon, Macalester College, Saint Paul, MN.*

What is the chromatic number of G_n, the graph whose vertices are the cells in a triangular grid of hexagons and whose edges correspond to two cells in the same row of adjacent cells in any of the three directions? The diagram shows G_7, with the neighbors of one vertex marked.



This graph can be viewed as the graph of moves of a rook in hexagonal chess (http://en.wikipedia.org/wiki/Hexagonal_chess).

1980. *Proposed by H. A. ShahAli, Tehran, Iran.*

For every S subset of the plane, let $\operatorname{diam}(S) = \sup\{\|x - y\| : x, y \in S\}$. Let $n \geq 1$ be an integer and $S_1, S_2, \ldots, S_n$ subsets of the plane such that $\sum_{k=1}^{n} \operatorname{diam}(S_k) < \sqrt{2}$. Define $S = \cup_{k=1}^{n} S_k$. Prove that there is a translation of S that avoids all points with integer coordinates. That is, prove that there are real numbers r and s such that $((r, s) + S) \cap (\mathbb{Z} \times \mathbb{Z}) = \emptyset$.

Quickies

Answers to the Quickies are on page 293.

Q1053. *Proposed by Herman Roelants, Center for Logic, Institute of Philosophy University of Leuven, Belgium.*

Determine all positive integers n for which

$$\frac{(n^2 + n - 1)^2}{2n + 1}$$

is an integer.

Q1054. *Proposed by Michael W. Botsko, Saint Vincent College, Latrobe, PA.*

Let $f : [a, b] \to \mathbb{R}$. The function f is said to have property (N) if f maps sets of (Lebesgue) measure zero into sets of measure zero. It is known that if f is differentiable on $[a, b]$, then f has property (N). If f is a derivative on $[a, b]$, is it necessary that f have property (N)?

Solutions

AM–GM inequality and a differential equation in the mix **June 2014**

1946. *Proposed by Marcel Chiriţă, Bucharest, Romania.*

Let $f : (0, \infty) \to (0, \infty)$ be a nondecreasing differentiable function. Characterize the functions $g : (0, \infty) \to (0, \infty)$ such that

$$2f(x)g(y) \le f(x)g(x) + f(y)g(y),$$

for all $x, y \in (0, \infty)$.

Solution by Eugene A. Herman, Grinnell College, Grinnell, IA.
The functions g are the positive multiples of f. Functions of the form $g = kf$ with $k > 0$ satisfy the requirements and the given inequality since $2ab \le a^2 + b^2$ for all $a, b \in \mathbb{R}$. Now suppose g satisfies the given inequality, and write the inequality in the form

$$\frac{1}{g(y)}(g(y) - g(x)) \le \frac{1}{f(x)}(f(y) - f(x)). \tag{1}$$

Interchanging x and y in (1), we obtain

$$\frac{1}{g(x)}(g(x) - g(y)) \le \frac{1}{f(y)}(f(x) - f(y)),$$

which implies that for all $x, y \in (0, \infty)$, we actually have

$$\frac{g(x)}{g(y)f(y)}(f(y) - f(x)) \le \frac{1}{g(y)}(g(y) - g(x)) \le \frac{1}{f(x)}(f(y) - f(x)). \tag{2}$$

For $y < x$, because f is assumed nondecreasing, $f(y) \le f(x)$ and so (1) implies $g(y) \le g(x)$. Hence, g is nondecreasing and then g has sided limits at every point. From (2), we conclude that g must be continuous by the squeeze theorem. Dividing (2) by $y - x$ ($y \ne x$), we see that g is in fact differentiable and $g'/g = f'/f$. Therefore, integrating this, we obtain $\ln g(x) = c + \ln f(x)$ for some real constant c, and finally $g(x) = e^c f(x)$ for all $x > 0$.

Editor's Note. The hypothesis f being nondecreasing is not needed. The function g must be locally bounded because if $g(y_n) \to \infty$ with $y_n \to y_0 > 0$, then (1) leads to $1 \le \frac{1}{f(x)}(f(y_0) - f(x))$, which is clearly false for $x = y_0$. This is enough to conclude from (2) that g is continuous. The rest of the argument follows as before.

Also solved by Michel Bataille (France), Elias Lampakis (Greece), Texas State University Problem Solvers Group, and the proposer. There were three incomplete or incorrect submissions.

A probabilistically expected inequality **June 2014**

1947. *Proposed by Raymond Mortini and Jérôme Noël, Université de Lorraine, Metz, France.*

Let n be a positive integer. Prove that

$$\sum_{k=0}^{n} |\cos k| \geq \frac{n}{2}.$$

Solution by Adnan Ali (student), A. E. C. S-4, Mumbai, India.
We know that $|\cos k| \leq 1$ implies $|\cos k| \geq \cos^2 k = (1+\cos 2k)/2$. Using the telescoping technique in conjunction with the formula

$$\cos 2k = \frac{\sin(2k+1) - \sin(2k-1)}{2\sin 1}, \quad k = 0, 1, 2 \ldots, n,$$

we obtain

$$\sum_{k=0}^{n} |\cos k| \geq \frac{n+1}{2} + \frac{1}{2}\sum_{k=0}^{n} \cos 2k = \frac{n+1}{2} + \frac{\sin(2n+1) + \sin 1}{4\sin 1}.$$

Then, by the MacLaurin series formula $\sin 1 = 1 - 1/3! + 1/5! + \cdots$, we know that, as an alternating series of decreasing terms, its limit lies between two consecutive partial sums, and so $1 > \sin 1 > 1 - 1/6 > 1/3$. This implies that

$$\sum_{k=0}^{n} |\cos k| \geq \frac{n}{2} + \frac{\sin(2n+1) + 3\sin 1}{4\sin 1} \geq \frac{n}{2} + \frac{\sin(2n+1) + 1}{4\sin 1} \geq \frac{n}{2}.$$

Editor's Note. Since the sequence $\{k\}_{k\in\mathbb{N}}$ is uniformly distributed in $[0,\pi]$, the average $\frac{1}{n}\sum_{k=0}^{n} |\cos k|$ converges to $\frac{1}{\pi}\int_0^{\pi} |\cos x| dx = \frac{2}{\pi} \approx 0.63662$, by Birkhoff's ergodic theorem. Then stronger inequalities can be obtained if n is chosen big enough. Tomas Persson and Mikael P. Sundqvist made these observations, and Eugene Herman showed that

$$\sum_{k=0}^{n} |\cos k| \geq \left(\frac{\sin 1 + \sin 2}{\pi}\right) n > \frac{n}{2}.$$

Also solved by Michael Bataille (France), Calvin Berry, John Christopher , L. Cíosoǵ, Robert L. Doucette, Eugene A. Herman, Elias Lampakis (Greece), Peter McPolin (Ireland), Missouri State University Problem Solving Group, Moubinool Omarjee (France), Paolo Perfetti (Italy), Kambiz Razminia (Iran), Amol Sasane (United Kingdom), Tomas Persson and Mikael P. Sundqvist (Sweden), Nora S. Thornber, Michael Vowe (Switzerland), and the proposer. There were six incomplete solutions.

A large power of $n!$ that divides $(mn)!$ **June 2014**

1948. *Proposed by Howard Carry Morris, Cordova, TN.*

For natural numbers m and n, it is known that $(mn)!$ is divisible by $(n!)^m$. Let $m = \max\{p^r : r \in \mathbb{N},\ p \text{ is prime, and } p^r \leq n\}$. Prove that $(mn)!$ is divisible by $(n!)^{m+1}$.

Solution by Daniel Fritze, Berlin, Germany.
Let q be any prime number not bigger than n, and let $s \in \mathbb{N}$ be the largest exponent of q such that $q^s \leq n$. The exponents of q in the prime factorizations of $(mn)!$ and $(n!)^{m+1}$ are given by

$$\sum_{i=1}^{t} \left\lfloor \frac{mn}{q^i} \right\rfloor, \text{ and } \sum_{i=1}^{s} \left\lfloor \frac{mn}{q^i} \right\rfloor,$$

where t is the largest exponent of q such that $q^t \leq mn$.

Obviously, $t \geq 2s$ since $m \geq q^s$ by the definition of m. Using this and the simple fact that $\lfloor ka \rfloor \geq \lfloor k \lfloor a \rfloor \rfloor = k\lfloor a \rfloor$ for $k \in \mathbb{N}$ and $a \in \mathbb{R}$, it follows that

$$\sum_{i=1}^{t} \left\lfloor \frac{mn}{q^i} \right\rfloor \geq \sum_{i=1}^{2s} \left\lfloor \frac{mn}{q^i} \right\rfloor = \sum_{i=1}^{s} \left\lfloor \frac{mn}{q^i} \right\rfloor + \sum_{i=s+1}^{2s} \left\lfloor \frac{mn}{q^i} \right\rfloor$$

$$\geq m \sum_{i=1}^{s} \left\lfloor \frac{n}{q^i} \right\rfloor + \sum_{i=s+1}^{2s} \left\lfloor \frac{q^s n}{q^i} \right\rfloor = (m+1) \sum_{i=1}^{s} \left\lfloor \frac{n}{q^i} \right\rfloor;$$

the conclusion follows because every prime that divides $n!$ is at most n.

Also solved by L. Cíosóg, Dmitry Fleischman, Joel Schlosberg, John H. Smith, and the proposer.

A midpoint-triangle of three similar triangles June 2014

1949. *Proposed by Rob Downes, Mountain Lakes High School, Mountain Lakes, NJ.*

Let $\triangle A_1B_1C_1$, $\triangle A_2B_2C_2$, and $\triangle A_3B_3C_3$ be triangles in the plane such that

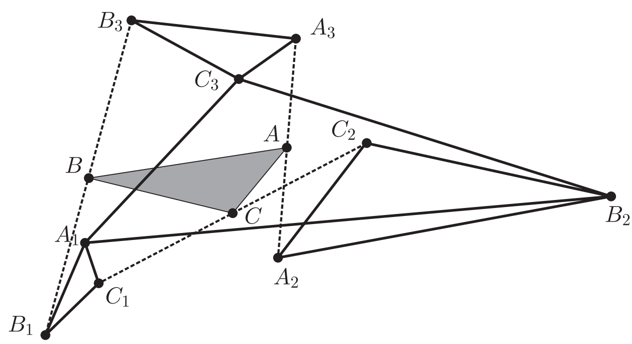
$$\triangle A_1B_1C_1 \sim \triangle A_2B_2C_2 \sim \triangle A_3B_3C_3.$$

Let A be the midpoint of $\overline{A_2A_3}$, B the midpoint of $\overline{B_1B_3}$, and C the midpoint of $\overline{C_1C_2}$. Prove that

$$\triangle ABC \sim \triangle A_1B_1C_1 \text{ or } A = B = C,$$

if and only if,

$$\triangle A_1B_2C_3 \sim \triangle A_1B_1C_1 \text{ or } A_1 = B_2 = C_3.$$



Solution by Joel Schlosberg, Bayside, NY.
We prove the statement using the additional condition that $\triangle A_1B_1C_1$, $\triangle A_2B_2C_2$, and $\triangle A_3B_3C_3$ are *directly* similar. (Otherwise, the points $A_1 = (0, 0)$, $B_1 = (1, 0)$, $C_1 = (0, 1)$, $A_2 = (2, 0)$, $B_2 = (3, 0)$, $C_2 = (2, 1)$, $A_3 = (0, 4)$, $B_3 = (1, 4)$, and $C_3 = (0, 3)$ form a counterexample to the original statement.)

Placing the triangles in the complex plane, use the lowercased label of a point to denote its corresponding complex number. For $\triangle XYZ$ we have that

$$\left|\frac{y-x}{z-x}\right| = \frac{XY}{XZ},$$

and $\arg(\frac{y-x}{z-x})$ corresponds uniquely to the signed angle $\angle YXZ$. So $\triangle X'Y'Z'$ is directly similar to $\triangle XYZ$ if and only if

$$\frac{y'-x'}{z'-x'} = \frac{y-x}{z-x}.$$

Thus, given $\triangle XYZ$ (which implicitly assumes that X, Y, and Z are all different), the points X', Y', and Z' satisfy that $\triangle X'Y'Z'$ is directly similar to $\triangle XYZ$ or $X' = Y' = Z'$ if and only if $(y' - x')(z - x) = (z' - x')(y - x)$. This last identity is equivalent to

$$\begin{aligned}0 &= (y'-x')(z-x) - (z'-x')(y-x) = x'(y-z) + y'(z-x) + z'(x-y)\\ &= (x', y', z') \cdot (y-z, z-x, x-y),\end{aligned}$$

where $\cdot$ denotes the dot product.
Let $v = (b_1 - c_1, c_1 - a_1, a_1 - b_1)$. Since $\triangle A_1B_1C_1 \sim \triangle A_2B_2C_2 \sim \triangle A_3B_3C_3$, then $(a_i, b_i, c_i) \cdot v = 0$ for $i \in \{1, 2, 3\}$. Note that

$$\begin{aligned}(a, b, c) &= \frac{1}{2}(a_2 + a_3, b_1 + b_3, c_1 + c_2)\\ &= \frac{1}{2}[(a_1, b_1, c_1) + (a_2, b_2, c_2) + (a_3, b_3, c_3) - (a_1, b_2, c_3)].\end{aligned}$$

It follows that, $\triangle ABC \sim \triangle A_1B_1C_1$ or $A = B = C$, if and only if

$$0 = (a, b, c) \cdot v = -\frac{1}{2}(a_1, b_2, c_3) \cdot v,$$

if and only if $\triangle A_1B_2C_3 \sim \triangle A_1B_1C_1$ or $A_1 = B_2 = C_3$.

Also solved by the proposer.

An asymmetric inequality **June 2014**

1950. *Proposed by Aleksandar Ilic, Facebook, Inc., Menlo Park, CA.*

Let $a_1 \geq a_2 \geq \cdots \geq a_n$ be nonnegative real numbers, such that $a_1 + a_2 + \cdots + a_n = 1$. Prove that

$$\sum_{1\leq i<j\leq n} a_i a_j (a_i + a_j)(2 - a_i - a_j) \leq \frac{8}{27},$$

and determine when equality is achieved.

Solution by Elias Lampakis, Kiparissia, Greece.
For $n = 2$ the arithmetic mean–geometric mean (AM-GM) inequality gives $a_1 a_2 \le (\frac{1}{2}(a_1 + a_2))^2 = \frac{1}{4} < \frac{8}{27}$. For $n = 3$, using the fact that $a_1 + a_2 + a_3 = 1$ we obtain

$$a_1a_2(1-a_3^2) + a_1a_3(1-a_2^2) + a_2a_3(1-a_1^2)$$
$$= a_1a_2 + a_1a_3 + a_2a_3 - a_1a_2a_3(a_1+a_2+a_3) = a_1a_2 + a_1a_3 + a_2a_3 - a_1a_2a_3$$
$$= (1-a_1)(1-a_2)(1-a_3).$$

By the AM-GM inequality,

$$(1-a_1)(1-a_2)(1-a_3) \le \left(\frac{3-a_1-a_2-a_3}{3}\right)^3 = \frac{8}{27},$$

with equality if and only if $a_1 = a_2 = a_3 = \frac{1}{3}$.
For $n \ge 4$, we first prove the following lemma.

Lemma 1. *Let $n \ge 4$ and $a_1 \ge a_2 \ge \cdots \ge a_n$ be nonnegative real numbers such that $a_1 + a_2 + \cdots + a_n = 1$. Then*

$$\sum_{3\le i<j\le n} (a_i^2 a_j + a_i a_j^2) + \sum_{3\le i<j<k\le n} 2a_ia_ja_k \le \sum_{1\le i<j<k<\ell\le n} 8a_ia_ja_ka_\ell.$$

Proof. Let $x \ge y \ge z \ge w \ge 0$ and $x + y + z + w = 1$. Then $\frac{1}{4} \le x$, $z + w \le 2y = 8 \cdot \frac{1}{4} \cdot y \le 8xy$, and multiplying by zw yields

$$z^2w + zw^2 \le 8xyzw. \tag{1}$$

If $n = 4$, then (1) gives the desired conclusion because the second sum on the left-hand side is empty. Now let $n \ge 5$. For $3 \le i < j \le n$, let $S = 1 - \sum_{k=3,k\ne i,j}^{n} a_k$, $x = a_1/S$, $y = a_2/S$, $z = a_i/S$, and $w = a_j/S$. Inequality (1) implies that

$$a_i^2a_j + a_ia_j^2 \le 8a_1a_2a_ia_j + \sum_{\substack{k=3\\k\ne i,j}}^{n} (a_i^2a_ja_k + a_ia_j^2a_k).$$

Then

$$\sum_{3\le i<j\le n} (a_i^2a_j + a_ia_j^2) \le \sum_{3\le i<j\le n} 8a_1a_2a_ia_j + \sum_{3\le i<j\le n} \sum_{\substack{k=3\\k\ne i,j}}^{n} (a_i^2a_ja_k + a_ia_j^2a_k)$$
$$= \sum_{3\le i<j\le n} 8a_1a_2a_ia_j + \sum_{3\le i<j<k\le n} 2(a_i^2a_ja_k + a_ia_j^2a_k + a_ia_ja_k^2). \tag{2}$$

For $3 \le i < j < k \le n$, we have that $2(a_i + a_j + a_k) \le 3(a_1 + a_2)$. It follows that $1 + a_i + a_j + a_k \le 4a_1 + 4a_2 + \sum_{\ell=3,\ell\ne i,j,k}^{n} a_\ell$. Thus,

$$\sum_{3\le i<j<k\le n} a_ia_ja_k(1 + a_i + a_j + a_k) \le 4(a_1 + a_2) \sum_{3\le i<j<k\le n} a_ia_ja_k +$$

$$\sum_{3\le i<j<k\le n}\left[a_ia_ja_k\sum_{\substack{\ell=3\\ \ell\ne i,j,k}}^{n}a_\ell\right]=4(a_1+a_2)\sum_{3\le i<j<k\le n}a_ia_ja_k+\sum_{3\le i<j<k<\ell\le n}4a_ia_ja_ka_\ell,$$

and so continuing from (2) we get

$$\begin{aligned}\sum_{3\le i<j\le n}(a_i^2a_j+a_ia_j^2)+\sum_{3\le i<j<k\le n}2a_ia_ja_k&\le\sum_{3\le i<j\le n}8a_1a_2a_ia_j\\ +\sum_{3\le i<j<k\le n}2a_ia_ja_k(1+a_i+a_j+a_k)&\le\sum_{3\le i<j\le n}8a_1a_2a_ia_j\\ +8(a_1+a_2)\sum_{3\le i<j<k\le n}a_ia_ja_k&+\sum_{3\le i<j<k<\ell\le n}8a_ia_ja_ka_\ell\\ &=\sum_{1\le i<j<k<\ell\le n}8a_ia_ja_ka_\ell,\end{aligned}$$

which proves the lemma.

We proceed to prove the problem's inequality for $n\ge 4$. Using that $\sum_{i=1}^n a_i=1$ we get

$$\begin{aligned}\sum_{1\le i<j\le n}a_ia_j(a_i+a_j)(2-a_i-a_j)&=\sum_{1\le i<j\le n}\left[(a_i^2a_j+a_ia_j^2)\left(1+\sum_{\substack{k=1\\ k\ne i,j}}^{n}a_k\right)\right]\\ &=\sum_{1\le i<j\le n}(a_i^2a_j+a_ia_j^2)+\sum_{1\le i<j\le n}\sum_{\substack{k=1\\ k\ne i,j}}^{n}(a_i^2a_ja_k+a_ia_j^2a_k)\\ &=\sum_{1\le i<j\le n}(a_i^2a_j+a_ia_j^2)+\sum_{1\le i<j<k\le n}2(a_i^2a_ja_k+a_ia_j^2a_k+a_ia_ja_k^2).\quad(3)\end{aligned}$$

Again, using that $\sum_{i=1}^n a_i=1$, we obtain

$$\sum_{1\le i<j<k\le n}2(a_i^2a_ja_k+a_ia_j^2a_k+a_ia_ja_k^2)=\sum_{1\le i<j<k\le n}2(a_ia_ja_k(a_i+a_j+a_k))=$$

$$\begin{aligned}\sum_{1\le i<j<k\le n}\left[2a_ia_ja_k\left(1-\sum_{\substack{\ell=1\\ \ell\ne i,j,k}}^{n}a_\ell\right)\right]&=\sum_{1\le i<j<k\le n}2a_ia_ja_k-\sum_{1\le i<j<k<\ell\le n}8a_ia_ja_ka_\ell\\ &\le\sum_{\substack{1\le i\le 2\\ i+1\le j<k\le n}}2a_ia_ja_k-\sum_{3\le i<j\le n}(a_i^2a_j+a_ia_j^2),\end{aligned}$$

where the last inequality is given by the lemma. Plugging this in Equation (3) gives

$$\sum_{1\le i<j\le n}a_ia_j(a_i+a_j)(2-a_i-a_j)\le\sum_{\substack{1\le i\le 2\\ i+1\le j<k\le n}}2a_ia_ja_k+\sum_{\substack{1\le i\le 2\\ i+1\le j\le n}}(a_i^2a_j+a_ia_j^2)$$

$$= (a_1 + a_2) \left(\sum_{\substack{j=1 \\ j \neq 2}}^{n} a_j \right) \left(\sum_{j=2}^{n} a_j \right) \leq \left(\frac{2(a_1 + a_2 + \cdots a_n)}{3} \right)^3 = \frac{8}{27},$$

where the last inequality is given by the AM-GM inequality. Equality holds if and only if $n \geq 3$, $a_1 = a_2 = a_3 = 1/3$, and $a_4 = a_5 = \cdots = a_n = 0$.

Editor's Note. Many submissions incorrectly stated that for $n \geq 4$ the maximum is attained when $a_1 = a_2 = \cdots = a_n = 1/n$.

Also solved by the proposer. There were four incorrect or incomplete submissions.

Answers

Solutions to the Quickies from page 286.

A1053. Let $N = (n^2 + n - 1)^2/(2n + 1)$. The equation $(n^2 + n - 1)^2 \equiv 0 \pmod{2n + 1}$ is equivalent to $(4n^2 + 4n - 4)^2 \equiv 0 \pmod{2n + 1}$ because $2n + 1$ is an odd integer. The solutions are given by $[(2n + 1)^2 - 5]^2 \equiv 0 \pmod{2n + 1}$, which is equivalent to $25 \equiv 0 \pmod{2n + 1}$. This implies that there are precisely two solutions in positive integers: $n = 2$ with $N = 5$, and $n = 12$ with $N = 961$.

A1054. The answer is no. To see this, let f be the Cantor ternary function on $[0, 1]$. Recall that f is continuous, increasing, $f'(x) = 0$ almost everywhere, $f(0) = 0$, and $f(1) = 1$. Since f is a continuous function, it is a derivative on $[0, 1]$. Suppose f has property (N). Since f is also continuous and of bounded variation on $[0, 1]$, by a well-known theorem of Banach and Zarecki, f is absolutely continuous on $[0, 1]$. Thus, by the fundamental theorem of Lebesgue integral calculus, $\int_0^1 f'(x)dx = f(1) - f(0)$. Since $f'(x) = 0$ almost everywhere, the previous equality yields $0 = 1$, a contradiction. Thus, f is a derivative that does not have property (N).

Saint Hubert is the patron saint of mathematicians. The next time your students, or you for that matter, need assistance with mathematics, try praying to our patron saint. Saint Hubert is also the saint to pray to if you suffer from a dog bite. Students may find it interesting to investigate how Saint Hubert became the patron saint of mathematicians and affiliated with dog bites. Somehow, I believe that many students will see an immediate connection between the two, as they have suffered from both!

REVIEWS

PAUL J. CAMPBELL, *Editor*
Beloit College

Assistant Editor: Eric S. Rosenthal, West Orange, NJ. Articles, books, and other materials are selected for this section to call attention to interesting mathematical exposition that occurs outside the mainstream of mathematics literature. Readers are invited to suggest items for review to the editors.

Casazza, Peter, Steven G. Krantz, and Randi D. Ruden, *I, Mathematician*, MAA, 2015; xiii + 273 pp, $50(P), $40 for MAA members. ISBN 978-0-88385-584-0.

This book is filled with delightful apologiae, from a multitude of perspectives, for mathematics, mathematicians, and the "mathematical life." Included are numerous autobiographical sketches and even the observations of a spouse of a mathematician. The title reminds me, as it will many mathematicians, of Isaac Asimov's *I, Robot* (1950), which announced the Three Laws of Robotics. Let me distill the contents of this newer book into Three Laws of Mathematicians: 1. Mathematicians have no choice but to be mathematicians (because they love mathematics so much). 2. Mathematicians are different—disconnected from reality—but (because of the First Law) they can't help it. 3. Mathematicians pursue an arcane art, appreciation of which is an acquired taste, a taste acquired by few (generally only by other mathematicians).

Roberts, Siobhan, *Genius at Play: The Curious Mind of John Horton Conway*, Bloomsbury, 2015; xxiv + 452 pp, $30. ISBN 978-1-62040-593-2. A life in games, *Quanta Magazine* `https://www.quantamagazine.org/20150828-john-conway-a-life-in-games/` (adapted from the book).

"Archimedes, Mick Jagger, Salvador Dalí, and Richard Feynman all rolled into one." Yup, that pretty much sums up John Horton Conway. This is the rare example of an interactive biography, assembled over 9 years, in which subject Conway collaborated intensively with biographer Roberts, going back and forth about just about everything. Much mathematics is explained or explored, and the book is very hard to put down, in part because Conway so vividly fulfills the First Law of Mathematicians by obviously loving mathematics so much.

Cook, Gareth, High(e)r power: The singular mathematical mind of Terry Tao, *New York Times Magazine* (26 July 2015) 44–49.

Here is a brief sketch about Fields Medalist Terence Tao, a mathematician who violates the Second Law of Mathematicians by being normal: "They will never make a movie about him." (Well, a bit more absent-minded than normal.) Moving on from work on the twin prime conjecture, Tao is now investigating the Navier–Stokes equations and why water doesn't explode.

Swanson, Ana, The mathematically proven winning strategy for 14 of the most popular games, *Washington Post* (8 May 2015), `http://www.washingtonpost.com/news/wonkblog/wp/2015/05/08/how-to-win-any-popular-game-according-to-data-scientists/`.

What games? Battleship, rock-paper-scissors, Scrabble, Monopoly, Pac-Man, coin toss, "The Price Is Right," Connect Four, Diplomacy, "Jeopardy!," poker, tic-tac-toe, and—chess (?!). Well for chess and most of the others, it's advantageous tips that are offered rather than a winning strategy. A few of the items (coin toss, tic-tac-toe, Connect Four) refer the reader to Numberphile videos.

Math. Mag. **88** (2015) 294-295. doi:10.4169/math.mag.88.4.294. © Mathematical Association of America

Mann, Casey, Jennifer McLoud, and David Von Derau, Type 15 convex pentagon, http://i.imgur.com/AyDuinZ.png .

Garrity, Mike, Type 15 convex pentagon, http://blogs.mathworks.com/graphics/2015/08/19/type-15-convex-pentagon/ .

Lamb, Evelyn J., There's something about pentagons, AMS Blogs: Blog on Math Blogs (7 September 2015), http://blogs.ams.org/blogonmathblogs/2015/09/07/theres-something-about-pentagons/ .

Pegg, Ed, Coding new pentagon tiling, http://community.wolfram.com/groups/-/m/t/550169 .

Three researchers (including an undergraduate) at University of Washington–Bothell have discovered the 15th known class of pentagon that can tile the plane. No one knows if there are more; the 14th was discovered in 1985. The pentagon has interior angles of 90, 150, 60, 135, and 105 degrees. Garrity gives Matlab code, and Pegg gives Mathematica code, that draws the tiling, while Lamb provides context and pointers to other announcements of the discovery.

Padgett, Jason, and Maureen Ann Seaberg, *Struck by Genius: How a Brain Injury Made Me a Mathematical Marvel*, Houghton Mifflin Harcourt, 2014; ix + 234 pp, $27. ISBN 978-0-544-04560-6.

Struck by genius. http://www.struckbygenius.com/.

Your friends and acquaintances may have heard of author Padgett, whose brain was injured in a violent robbery of him in 2002. He refers to himself now as an "acquired savant" with conceptual synesthesia. Unlike perceptual synesthetes, who see colors associated with numbers and letters, Padgett sees shapes. Apart from a fascination with geometric shapes and drawings of geometric approximations to pi, Padgett does not claim any of the abilities (e.g., exhaustive memory, calculational expertise) associated with what are now known as autistic savants.

Numberphile. http://www.numberphile.com . Videos by Brady Haran.

This website offers about 200 video clips, most 5 to 10 minutes long, about numbers, probability, and mathematical concepts. The videos consist of interviews with mathematicians explaining ideas (e.g., such "stars" as Barry Mazur on $\sqrt{-15}$, Persi Diaconis on shuffling and coin-tossing), supplemented by graphics. There's lots of fun here, too: What would it be like to golf or play baseball in a hyperbolic universe? Will your students be convinced by the animated 48-second Monty Hall Problem Express Explanation? The videos are housed at YouTube and include some advertising (which can be clicked away); it's not easy to navigate to some of the "extra footage" clips. Although Numberphile notes support by the Mathematical Sciences Research Institute (MSRI), the MSRI site (regrettably) does not seem to link to Numberphile.

Berger, Arno, and Theodore P. Hill, *An Introduction to Benford's Law*, Princeton University Press, 2015; viii + 248 pp, $75. ISBN 978-0-691-16306-2.

Miller, Steven J. (ed.), *Benford's Law: Theory and Applications*, Princeton University Press, 2015; xxvi + 438 pp, $75. ISBN 978-0-691-14761-1.

Benford's law, also known as the first-digit phenomenon, refers to the observation (originally by Simon Newcomb) that in many datasets, the first significant digits are not uniformly distributed but instead tend to follow $P(d) = \log_{10}(1 + \frac{1}{d})$. For example, in such datasets we have $P(1) \approx 0.3$. What datasets this "law" applies to, and what the mathematical justification could be, is the subject of both of these books. In both, the mathematics is mostly advanced; but Miller includes considerably more less-technical material about applications of Benford's law. Those include detecting fraud in accounting, voting, and government statistics, as well as uses in psychology and imaging.

NEWS AND LETTERS

44^{th} United States of America Mathematical Olympiad and 6^{th} United States of America Junior Mathematical Olympiad

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The United States of America Mathematics Olympiad (USAMO) and Junior Olympiad (USAJMO) are high-level, Olympiad-style contests offered by the Committtee on American Mathematics Competitions of the Mathematical Association of America. The two competitions are administered simultaneously, this year on April 28 and 29. Both competitions consist of 3 problems for each of 2 days, with an allowed time of 4.5 hours each day. This is the format used by the International Mathematical Olympiad (IMO).

The USAMO is used to select a team of six students to represent the nation in the IMO, and the level of the problems reflects the level expected on the IMO competition. The USAJMO, offered to students in grade 10 and below, is used to identify students to train for participation in future IMO competitions. In setting problems for this competition, the Committee strives to provide a nicely balanced link between the computational character of the AIME problems and the more advanced proof-oriented problems of the USAMO.

The two contests included three common problems. On the first day problems JMO2 and JMO3 were the same as USAMO 1 and USAMO 2, respectively, and on the second day JMO6 and USAMO 4 were identical.

This year 294 students sat for the USAMO contest, and 251 for the USAJMO. More information is available on the AMC section of the MAA website.

USAMO Problems

1. Solve in integers the equation

$$x^2 + xy + y^2 = \left(\frac{x+y}{3} + 1\right)^3.$$

2. Quadrilateral $APBQ$ is inscribed in circle ω with $\angle P = \angle Q = 90^\circ$ and $AP = AQ < BP$. Let X be a variable point on segment $\overline{PQ}$. Line AX meets ω again at S (other than A). Point T lies on arc AQB of ω such that $\overline{XT}$ is perpendicular to $\overline{AX}$. Let M

Math. Mag. **88** (2015) 296–304. doi:10.4169/math.mag.88.4.296.

denote the midpoint of chord $\overline{ST}$. As X varies on segment $\overline{PQ}$, show that M moves along a circle.

3. Let $S = \{1, 2, \ldots, n\}$, where $n \geq 1$. Each of the 2^n subsets of S is to be colored red or blue. (The subset itself is assigned a color and not its individual elements.) For any set $T \subseteq S$, we then write $f(T)$ for the number of subsets of T that are blue.

   Determine the number of colorings that satisfy the following condition: for any subsets T_1 and T_2 of S,

$$f(T_1)f(T_2) = f(T_1 \cup T_2)f(T_1 \cap T_2).$$

4. Steve is piling $m \geq 1$ indistinguishable stones on the squares of an $n \times n$ grid. Each square can have an arbitrarily high pile of stones. After he is finished piling his stones in some manner, he can then perform *stone moves*, defined as follows. Consider any four grid squares, which are corners of a rectangle, i.e., in positions (i, k), (i, l), (j, k), (j, l) for some $1 \leq i, j, k, l \leq n$, such that $i < j$ and $k < l$. A stone move consists of either removing one stone from each of (i, k) and (j, l) and moving them to (i, l) and (j, k), respectively, or removing one stone from each of (i, l) and (j, k) and moving them to (i, k) and (j, l), respectively.

   Two ways of piling the stones are equivalent if they can be obtained from one another by a sequence of stone moves.

   In how many different nonequivalent ways can Steve pile the stones on the grid?

5. Let a, b, c, d, e be distinct positive integers such that $a^4 + b^4 = c^4 + d^4 = e^5$. Show that $ac + bd$ is a composite number.

6. Consider $0 < \lambda < 1$, and let A be a multiset of positive integers. Let $A_n = \{a \in A : a \leq n\}$. Assume that for every $n \in \mathbb{N}$, the set A_n contains at most $n\lambda$ numbers. Show that there are infinitely many $n \in \mathbb{N}$ for which the sum of the elements in A_n is at most $\frac{n(n+1)}{2}\lambda$.
   (A multiset is a set-like collection of elements in which order is ignored, but repetition of elements is allowed and multiplicity of elements is significant. For example, multisets $\{1, 2, 3\}$ and $\{2, 1, 3\}$ are equivalent, but $\{1, 1, 2, 3\}$ and $\{1, 2, 3\}$ differ.)

SOLUTIONS

1. Let $x + y = 3k$, with $k \in \mathbb{Z}$. Then $x^2 + x(3k - x) + (3k - x)^2 = (k + 1)^3$, which reduces to

$$x^2 - (3k)x - (k^3 - 6k^2 + 3k + 1) = 0.$$

   Its discriminant Δ is

$$9k^2 + 4(k^3 - 6k^2 + 3k + 1) = 4k^3 - 15k^2 + 12k + 4.$$

   We notice the (double) root $k = 2$, so $\Delta = (4k + 1)(k - 2)^2$. It follows that $4k + 1 = (2t + 1)^2$ for some nonnegative integer t, hence $k = t^2 + t$ and

$$x = \frac{1}{2}(3(t^2 + t) \pm (2t + 1)(t^2 + t - 2)).$$

   We obtain $(x, y) = (t^3 + 3t^2 - 1, -t^3 + 3t + 1)$ and $(x, y) = (-t^3 + 3t + 1, t^3 + 3t^2 - 1)$, $t \in \{0, 1, 2, \ldots\}$.

   This problem and solution were suggested by Titu Andreescu.

2. Let O denote the center of ω, and let W denote the midpoint of segment $\overline{AO}$. Denote by Ω the circle centered at W with radius WP. We will show that $WM = WP$, which will imply that M always lies on Ω and so solve the problem. Set r to be the radius of circle ω. Applying the median formula in triangles APO, SWT, ASO, ATO gives

$$4WP^2 = 2AP^2 + 2OP^2 - AO^2 = 2AP^2 + r^2,$$
$$4WM^2 = 2WS^2 + 2WT^2 - ST^2,$$
$$2WS^2 = AS^2 + OS^2 - AO^2/2 = AS^2 + r^2/2,$$
$$2WT^2 = AT^2 + OT^2 - AO^2/2 = AT^2 + r^2/2.$$

Adding the last three equations yields $4WM^2 = AS^2 + AT^2 - ST^2 + r^2$. It suffices to show that

$$4WP^2 = 4WM^2 \quad \text{or} \quad AS^2 + AT^2 - ST^2 = 2AP^2. \tag{1}$$

Because $\overline{XT} \perp \overline{AS}$,

$$\begin{aligned} AT^2 - ST^2 &= (AX^2 + XT^2) - (SX^2 + XT^2) \\ &= AX^2 - SX^2 \\ &= (AX + XS)(AX - XS) \\ &= AS(AX - XS). \end{aligned}$$

It follows that $AS^2 + AT^2 - ST^2 = AS^2 + AS \cdot (AX - XS) = AS^2 + AS(2AX - AS) = 2AS \cdot AX$, and (1) reduces to $AP^2 = AS \cdot AX$, which is true because triangle APX is similar to triangle ASP (as $\angle PAX = \angle SAP$ and $\angle APX = \text{arc}(AQ)/2 = \text{arc}(AP)/2 = \angle ASP$).

This problem and solution were suggested by Zuming Feng and Jacek Fabrykowski.

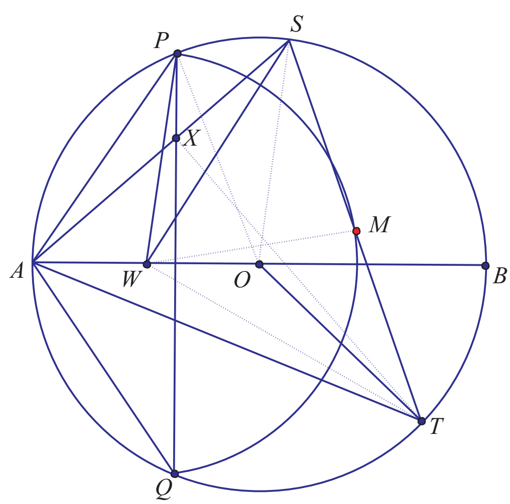

Figure 1

3. Specifically, the colorings we want are of the following forms: either there are no blue sets; or for each element $x \in S$ we define one of three types of restriction—either x must be in T, x can't be in T, or x is unrestricted—and the blue sets T are exactly the ones that satisfy every restriction. It's easy to check such a coloring meets the condition, using the formula

$$f(T) = \prod_{x \in T} a_x \prod_{x \notin T} b_x,$$

where $a_x = 2$ if x is unrestricted and 1 otherwise, and $b_x = 0$ if x is required to be present and 1 otherwise.

We want to show that if there's at least one blue set, then the class of blue sets is of this form.

If some element of S is in every blue set, take it out and use induction. If some element of S is not in any blue set, take it out and use induction. Otherwise, every element x has some blue set containing it and some blue set not containing it. In this case we'll show that all sets are blue (i.e., every element is unrestricted).

First show $\emptyset$ is blue. To show this, let T be a minimal blue set. If T is nonempty, take $x \in T$; by assumption there's blue T' not containing x. Then the condition is violated with T and T', since $f(T \cap T') = 0$. Next we show any singleton is blue. Otherwise, let U be a minimal blue set containing x, and let $T = \{x\}$ and $T' = U \setminus \{x\}$. We get $1 \cdot m = 1 \cdot (1 + m)$ (where $m = f(T')$), a contradiction. Finally, any set is blue. Otherwise, let U be a minimal nonblue set and x, y two different elements. Taking $T = U \setminus \{x\}$, $T' = U \setminus \{y\}$ gives a contradiction. We conclude that the number of required coloring is $3^n + 1$.

This problem and solution were suggested by Gabriel Carroll.

4. We think of the pilings as assigning a positive integer to each square on the grid. We restrict ourselves to the types of moves in which we take a lower left and upper right stone and move them to the upper left and lower right of our chosen rectangle. Call this a Type 1 stone move. We claim that we can perform a sequence of Type 1 stone moves on any piling to obtain an equivalent piling for which we cannot perform any Type 1 move, i.e., in which no square that has stones is above and to the right of any other square that has stones. We call such a piling a "down-right" piling.

To prove that any piling is equivalent to a down-right piling, first consider the squares in the leftmost column and topmost row of the grid. Let a be the entry (number of stones) in the upper left corner, and let b and c be the sum of the remaining entries in the leftmost column and topmost row, respectively. If $b < c$, we can perform a sequence of Type 1 stone moves to remove all the stones from the leftmost column except for the top entry, and if $c < b$, we can similarly clear all squares in the top row except for the top left square. In the former case, we can now ignore the leftmost column and repeat the process on the second-to-leftmost column and the top row. Similarly, in the latter case, we can ignore the top row and proceed as before. Since the corner square a cannot be part of any Type 1 move at each step in the process, it follows that we end up with a down-right piling.

We next show that down-right pilings in any size grid (not necessarily $n \times n$) are uniquely determined by their row sums and column sums, given that the row sums and column sums are nonnegative integers which sum to m both along the rows and the columns. Let the topmost row sum be R_1 and the leftmost column sum be C_1. Then the upper left square must contain $\min(R_1, C_1)$ stones, since otherwise there would be stones both in the first row and first column that are not in the upper left square. The one that is smaller indicates that either the row or the column, respectively, is empty save for the upper left square. Then we can remove this row or column and are reduced to a smaller grid in which we know all the row and column sums. Since one-row and one-column pilings are clearly uniquely determined by their column and row sums, it follows by induction that down-right pilings are determined uniquely by their row sums and column sums.

Finally, notice that row sums and column sums are both invariant under stone moves. Therefore, every piling is equivalent to a *unique* down-right piling. It therefore suffices to count the number of down-right pilings, which is also equivalent to counting the number of possibilities for the row sums and column sums. As stated above, the row sums and column sums can be the sums of any two n-tuples of nonnegative integers that each sum to m. The number of such tuples is $\binom{n+m-1}{m}$, and so the total number of nonequivalent pilings is the number of pairs of these tuples, i.e., $\left(\binom{n+m-1}{m}\right)^2$.

This problem and solution were suggested by Maria Monks.

5. We approach the proof indirectly by assuming that $p = ac + bd$ is a prime. By symmetry, we may assume that $\max\{a, b, c, d\} = a$, then because $a^4 + b^4 = c^4 + d^4$, we infer that $\min\{a, b, c, d\} = b$. Note that $ac \equiv -bd \pmod{p}$, implying that $a^4c^4 \equiv b^4d^4 \pmod{p}$. Consequently, we have

$$b^4d^4 + b^4c^4 \equiv a^4c^4 + b^4c^4 = c^8 + c^4d^4 \pmod{p},$$

from which it follows that $(c^4 + d^4)(b^4 - c^4) \equiv 0 \pmod{p}$. Thus, p divides at least one of $b - c, b + c, b^2 + c^2, c^4 + d^4$. Because $p = ac + bd > c^2 + b^2$, and $-(b^2 + c^2) < b - c < 0$ (because b and c are distinct), p must divide $c^4 + d^4 = e^5$. Thus $p^5 = (ac + bd)^5$ divides $c^4 + d^4$, which is clearly impossible because it is evident that $(ac + bd)^5 > c^4 + d^4$.

This problem and solution were suggested by Mohsen Jamaali.

6. Set $b_n = |A_n|$, $a_n = n\lambda - |A_n| \geq 0$. We have $b_i - b_{i-1}$ elements in A equal to i, therefore the sum of elements in A_n is

$$n(b_n - b_{n-1}) + (n-1)(b_{n-1} - b_{n-2}) + \cdots = nb_n - (b_1 + b_2 + \cdots + b_{n-1}).$$

If we recall that $b_n = n\lambda - a_n$, the sum of elements in A_n can be written as

$$\Sigma_n = \lambda\frac{n(n+1)}{2} - (na_n - a_{n-1} - \cdots - a_1).$$

Assume that for all $n \geq n_0$ the sum of elements in A_n is greater that $\lambda\frac{n(n+1)}{2}$. We deduce thus that $na_n < a_{n-1} + a_{n-2} + \cdots + a_1$, so

$$a_n < \frac{a_{n-1} + a_{n-2} + \cdots + a_1}{n} < \frac{a_1 + a_2 + \cdots + a_{n-1}}{n-1}.$$

Set $M_n = \max\{a_1, a_2, \ldots, a_n\}$. It follows that $a_n \leq \frac{(n-1)M_n}{n} < M_n$, so $M_{n+1} = M_n = M$, where $M = M_{n_0}$. However, we note that $\{a_{k+1} - a_k\} = \lambda$ so $\{(M - a_k) - (M - a_{k+1})\} = \lambda$ and since $M - a_k$, $M - a_{k+1} \geq 0$ we deduce that $M - a_k + M - a_{k+1} \geq \min\{\lambda, 1 - \lambda\}$. To see this, note that if $a_k > a_{k+1}$, then $(M - a_k) - (M - a_{k+1}) \leq \lambda - 1$, thus $M - a_{k+1} \geq 1 - \lambda$, and if $a_k < a_{k+1}$, then $(M - a_k) - (M - a_{k+1}) \geq \lambda$, so $M - a_{k+1} \geq \lambda$. We deduce from here that $a_k + a_{k+1} \leq 2M - \lambda$ and from here we can easily deduce $a_n \leq M - \frac{\lambda}{2}$. Thus starting from $n = n_0 + 1$, we get $a_n \leq M - \frac{\lambda}{2}$. Now let $u = \frac{\lambda}{3}$.

We claim by induction on k that for $n \geq n_k$ we have $a_n \leq M - ku$. The basis is already proven. Now we can prove that for $k > n_m$ we have $a_k + a_{k+1} \leq 2M - 2mu - \lambda = 2M - (2m+3)u$. Thus we may take $n > (m+1)n_m$ and deduce that

$$a_n \leq \frac{n_m(M - 2mu) + (n - n_m)(M - (m + \frac{3}{2})u)}{n} \leq M - (m+1)u$$

and the induction is done. However, for $m > \frac{M}{u}$ we get $a_n < 0$, a contradiction.

This problem and solution were suggested by Iurie Boreico.

USAJMO Problems

1. Given a sequence of real numbers, a move consists of choosing two terms and replacing each by their arithmetic mean. Show that there exists a sequence of 2015 distinct real numbers such that after one initial move is applied to the sequence—no matter what move—there is always a way to continue with a finite sequence of moves so as to obtain in the end a constant sequence.
2. Same as USAMO 1.
3. Same as USAMO 2.
4. Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that

$$f(x) + f(t) = f(y) + f(z)$$

for all rational numbers $x < y < z < t$ that form an arithmetic progression. ($\mathbb{Q}$ is the set of all rational numbers.)

5. Let $ABCD$ be a cyclic quadrilateral. Prove that there exists a point X on segment $\overline{BD}$ such that $\angle BAC = \angle XAD$ and $\angle BCA = \angle XCD$ if and only if there exists a point Y on segment $\overline{AC}$ such that $\angle CBD = \angle YBA$ and $\angle CDB = \angle YDA$.
6. Same as USAMO 4.

Solutions

1. The sequence $(x_1, x_2, \ldots, x_{2015}) = (1, 2, \ldots, 2015)$ satisfies the required property (as does any arithmetic sequence).

Assume that $(x_m, x_n) = (m, n)$ is replaced by $\left(\frac{m+n}{2}, \frac{m+n}{2}\right)$ in the first move. We consider two cases.

In the first case we assume that none of m and n is equal to 1008. In the second move we replace $(x_{2016-m}, x_{2016-n}) = (2016 - m, 2016 - n)$ by $\left(2016 - \frac{m+n}{2}, 2016 - \frac{m+n}{2}\right)$. Let all the subsequent moves be applied to the pairs (x_j, x_{2016-j}), $j = 1, 2, \ldots, 1008$. This yields the constant sequence $(1008, 1008, \ldots, 1008)$.

In the second case we assume that one of m and n, say n, is equal to 1008. After the first move we have $x_m = x_{1008} = \frac{1008+m}{2}$. Choose k different from 1008, m, and $2016 - m$. We illustrate our next four moves in the following table. (In each move, we operate on the numbers in bold.)

$$
\begin{aligned}
&(x_k,\ x_m,\ x_{1008},\ x_{2016-m},\ x_{2016-k})\\
=\ &\left(\boldsymbol{k},\ \frac{1008+m}{2},\ \frac{1008+m}{2},\ 2016-m,\ \mathbf{2016}-\boldsymbol{k}\right)\\
\to\ &\left(\mathbf{1008},\ \frac{1008+m}{2},\ \frac{1008+m}{2},\ \mathbf{2016}-\boldsymbol{m},\ 1008\right)\\
\to\ &\left(\frac{\mathbf{3024}-\boldsymbol{m}}{\mathbf{2}},\ \frac{\mathbf{1008}+\boldsymbol{m}}{\mathbf{2}},\ \frac{1008+m}{2},\ \frac{3024-m}{2},\ 1008\right)\\
\to\ &\left(1008,\ 1008,\ \frac{\mathbf{1008}+\boldsymbol{m}}{\mathbf{2}},\ \frac{\mathbf{3024}-\boldsymbol{m}}{\mathbf{2}},\ 1008\right)\\
\to\ &(1008,\ 1008,\ 1008,\ 1008,\ 1008)
\end{aligned}
$$

Finally, we apply the move to all the pairs (x_j, x_{2016-j}) (with $j \neq m, k, 2016-m, 2016-k$) to obtain the constant sequence $(1008, 1008, \ldots, 1008)$.

This problem and solution were suggested by Razvan Gelca.

2. Same as USAMO 1.
3. Same as USAMO 2.
4. Choose any $n \in \mathbb{Z}$, $t \in \mathbb{Q}$. Applying the condition for $nt, (n+1)t, (n+2)t, (n+3)t$ yields

$$f((n+3)t) - f((n+2)t) = f((n+1)t) - f(nt)$$

and similarly

$$f((n+4)t) - f((n+3)t) = f((n+2)t) - f((n+1)t).$$

Adding the two yields

$$f((n+4)t) - f((n+2)t) = f((n+2)t) - f(nt),$$

in particular $f(2kt+2t) - f(2kt)$ is the same for all $k \in \mathbb{Z}$, which means f is linear on $2t \cdot \mathbb{Z}$. Since $\mathbb{Q}$ is a nested union of such sets, f is linear and all linear functions work.

This problem and solution were suggested by Iurie Boreico.

5. By the symmetry, it suffices to show the "only if" part by assuming that there exists a point X on segment $\overline{BD}$ such that $\angle BAC = \angle XAD$ and $\angle BCA = \angle XCD$.

Because $ABCD$ is cyclic, we have $\angle XAD = \angle BAC = \angle BDC = \angle XDC$ and $\angle XDA = \angle BDA = \angle BCA = \angle XCD$. Hence triangles AXD and DXC (and ABC) are similar to each other. In particular,

$$\frac{AX}{DX} = \frac{DX}{XC} \quad \text{or} \quad DX^2 = AX \cdot CX.$$

Because $\angle BAC = \angle XAD$, we have $\angle BAX = \angle CAD$. Because $ABCD$ is cyclic, we have $\angle CAD = \angle CBD = \angle CBX$. Consequently, $\angle BAX = \angle CBX$. Note that

$$\angle AXB = \angle XAD + \angle ADX = \angle BAC + \angle ACB = \angle BDC + \angle DCX = \angle CXB.$$

From the above facts, we conclude that triangles ABX and BCX (and ACD) are similar to each other and so we have $BX^2 = AX \cdot CX$. Thus, $BX^2 = AX \cdot CX = DX^2$; that is, X is the midpoint of the segment $\overline{BD}$. Therefore,

$$\frac{AB}{BC} = \frac{DX}{XC} = \frac{BX}{XC} = \frac{AD}{DC} \quad \text{or} \quad \frac{BC}{CD} = \frac{BA}{AD}.$$

Construct point Y on segment $\overline{AC}$ such that $\angle CBD = \angle YBA$. From $\angle CBD = \angle YBA$ and $\angle BAY = \angle BAC = \angle BDC$ we conclude that triangles BAY and BDC are similar to each other, from which it follow that

$$\frac{BY}{YA} = \frac{BC}{CD} = \frac{BA}{AD} \quad \text{or} \quad \frac{BY}{BA} = \frac{AY}{AD}.$$

Note also that $\angle YBA = \angle CBD = \angle CAD = \angle YAD$. We conclude that triangles BYA and AYD are similar to each other, implying that $\angle CDB = \angle YAB = \angle YDA$. This is the desired point Y.

By symmetry, it suffices to show that there exists X on the segment $\overline{BD}$ such that $\angle BAC = \angle XAD$ and $\angle BCA = \angle XCD$ if and only if $AB \cdot CD = AD \cdot BC$.

There is a unique point X_1 on segment $\overline{BD}$ such that $\angle X_1AD = \angle BAC$. There is a unique point X_2 on segment $\overline{BD}$ such that $\angle BCA = \angle X_2CD$. Because $ABCD$ is cyclic, $\angle BCA = \angle BDA = \angle X_1DA$. Hence triangles ABC and AX_1D are similar to each other, implying that

$$\frac{AC}{BC} = \frac{AD}{X_1D}.$$

Likewise, we can show that ABC and DX_2C are similar to each other and $\frac{AB}{AC} = \frac{DX_2}{DC}$. Multiplying the last two equations together gives

$$\frac{AB}{BC} = \frac{AB}{AC} \cdot \frac{AC}{BC} = \frac{DX_2}{DC} \cdot \frac{AD}{X_1D},$$

from which it follows that

$$\frac{AB \cdot CD}{AD \cdot BC} = \frac{DX_2}{DX_1}.$$

Note that point X exists if and only if $X_1 = X_2$, or $DX_2 = DX_1$; that is, $AB \cdot CD = AD \cdot BC$.

This problem and solution were suggested by Sungyoon Kim.

6. Same as USAMO 4.

The top twelve students on the 2015 USAMO were (in alphabetical order):

Ryan Alweiss	12	Bergen County Academies	NJ
Kritkorn Karntikoon	12	Loomis Chaffee School	CT
Michael Kural	11	Greenwich High School	CT
Celine Liang	11	Saratoga High School	CA
Allen Liu	11	Penfield Senior High School	NY
Yang Liu	12	Ladue Horton Watkins High School	MO
Shyam Narayanan	12	Blue Valley West High School	KS
Kevin Ren	9	Torrey Pines High School	CA
Zhuoqun Song	12	Phillips Exeter Academy	NH
David Stoner	12	South Aiken High School	SC
Kevin Sun	11	Phillips Exeter Academy	NH
Danielle Wang	12	Stanford Math Circle/Stanford University	CA

The top fifteen students on the 2015 USAJMO were (in alphabetical order):

Hongyi Chen	9	Fairview High School	CO
Daniel Chiu	10	Catlin Gabel School	OR
Vincent Huang	8	Schimelpfenig Middle School	TX
Yuan Liao	10	Sharon High School	MA
Edwin Peng	9	Monta Vista High School	CA
Laura Pierson	10	College Preparatory School	CA
Nathan Remesh	8	William Diamond Middle	MA
Brian Reinhart	9	Oxbridge Academy Of The Palm Beaches	FL
Michael Ren	9	Phillips Academy	MA
Colin Tang	8	Interlake High School	WA
Zhou Tao	10	Quail Valley Middle School	TX
Vinjai Vale	9	Phillips Exeter Academy	NH
Brandon Wang	7	Saratoga High School	CA
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56^{th} International Mathematical Olympiad

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Problems (Day 1)

1. We say that a finite set $\mathcal{S}$ of points in the plane is *balanced* if, for any two different points A and B in $\mathcal{S}$, there is a point C in $\mathcal{S}$ such that $AC = BC$. We say that $\mathcal{S}$ is *center free* if for any distinct points A, B, and C in $\mathcal{S}$, there is no point P in $\mathcal{S}$ such that $PA = PB = PC$.

   (a) Show that for all integers $n \geq 3$, there exists a balanced set consisting of n points.

   (b) Determine all integers $n \geq 3$ for which there exists a balanced center-free set consisting of n points.

2. Determine all triples (a, b, c) of positive integers such that each of the numbers

$$ab - c, \quad bc - a, \quad ca - b$$

   is a power of 2. (A power of 2 is an integer of the form 2^n, where n is a nonnegative integer.)

3. Let ABC be an acute triangle with $AB > AC$. Let Γ be its circumcircle, H its orthocenter, and F the foot of the altitude from A. Let M be the midpoint of BC. Let Q be the point on Γ such that $\angle HQA = 90^\circ$, and let K be the point on Γ such that $\angle HKQ = 90^\circ$. Assume that the points A, B, C, K, and Q are all different and lie on Γ in this order.

   Prove that the circumcircles of triangles KQH and FKM are tangent to each other.

Problems (Day 2)

4. Triangle ABC has circumcircle Ω and circumcenter O. A circle Γ with center A intersects the segment BC at points D and E, such that B, D, E, and C are all different and lie on line BC in this order. Let F and G be the points of intersection of Γ and Ω, such that A, F, B, C, and G lie on Ω in this order. Let K be the second point of intersection of the circumcircle of triangle BDF and the segment AB. Let L be the second point of intersection of the circumcircle of triangle CGE and the segment CA.

   Suppose that the lines FK and GL are different and intersect at the point X. Prove that X lies on the line AO.

5. Let $\mathbb{R}$ denote the set of real numbers. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that:

$$f(x + f(x + y)) + f(xy) = x + f(x + y) = yf(x).$$

Math. Mag. **88** (2015) 305–315. doi:10.4169/math.mag.88.4.305.

6. The sequence $a_1, a_2, \ldots$ of integers satisfies the conditions:

   (i) $1 \le a_j \le 2015$ for all $j \ge 1$,

   (ii) $k + a_k \neq \ell + a_\ell$ for all $1 \le k < \ell$.

   Prove that there exist two positive integers b and N for which

$$\left| \sum_{j=m+1}^{n} (a_j - b) \right| \le 1007^2$$

   for all integers m and n such that $n > m \ge N$.

Solutions There are six problems and six members of the USA International Mathematical Olympiad team. In honor of our students' historic performance, we have invited each of them to submit their solution and perspective, modified and polished with hindsight.

1. (Solution written by Shyam Narayanan.)

   If n is odd, consider a regular n-gon. Note that if we label the n-gon's points $a_1, a_2, \ldots, a_n$, we have that, for any two points a_i, a_j, if $i + j$ is even, then $a_{\frac{(i+j)}{2}}$ is equidistant from a_i and a_j. But if $i + j$ is odd, then $i + j + n$ is even and $a_{\frac{(i+j+n)}{2}}$ is equidistant from a_i and a_j, where we take indices modulo n. Thus, a regular n-gon is balanced whenever n is odd. In fact, it is also center free because for any three distinct points a_i, a_j, a_k, if $Pa_i = Pa_j = Pa_k$, then P must be the circumcenter of the three points, i.e., the circumcenter of the n-gon, which is not in the set.

   Now, assume n is even. Consider instead the n points located at the following complex numbers (when plotted in the plane with the real axis horizontal and the imaginary axis vertical):

$$0, \quad 1 = e^{\frac{0i\pi}{3(n-2)}}, \quad e^{\frac{2i\pi}{3(n-2)}}, \quad e^{\frac{4i\pi}{3(n-2)}}, \quad \ldots, \quad e^{\frac{2(n-2)i\pi}{3(n-2)}}.$$

   Note that if we were to pick two of the $e^{\frac{2ki\pi}{3(n-2)}}$ points, they would both be equidistant from 0. Otherwise, if we picked 0 and $e^{\frac{2ki\pi}{3(n-2)}}$, if $k \le \frac{n-2}{2}$, they would both be equidistant from $e^{\frac{(2k+n-2)i\pi}{3(n-2)}}$, and if $k \ge \frac{n-2}{2}$, they would both be equidistant from $e^{\frac{(2k-n+2)i\pi}{3(n-2)}}$. Thus, this is a construction of a *balanced* set when n is even.

   For part (b), we have already proven that there is a *balanced, center-free* set with n points when n is odd. We prove there does not exist one when n is even.

   Suppose that for an even n, there exists a *balanced, center-free* set with n points. Then label the points $a_1, a_2, \ldots, a_n$. Also, suppose that for $i < j \le n$, b_{ij} is the a_k with the smallest index such that a_k is equidistant from a_i and a_j. Note that because the set is *balanced*, b_{ij} always exists, and because there are $\frac{(n)(n-1)}{2}$ ways to choose i, j, but only n ways to choose a_k, the pigeonhole principle implies that some a_k must be b_{ij} for at least $\frac{n-1}{2}$ distinct pairs i, j. In fact, because $n - 1$ is odd, some a_k must be b_{ij} for at least $\frac{n}{2}$ distinct pairs i, j.

   But unfortunately, this means that either k equals i or j for one of the pairs, which is clearly not possible, or by the pigeonhole principle, some element must be in more than one of the ordered pairs. But this means that a_k is in fact equidistant from three elements a_i, a_j, a_l, so the set is not *center free*.

Thus, there exists a *balanced* set consisting of n points for all $n \geq 3$, and there exists a *balanced, center-free* set with n points if and only if n is odd. ■

This problem was proposed by Netherlands.

2. (Solution written by Yang Liu.)

It is easy to see that all of a, b, c are greater than 1.

For a positive integer n, define $\upsilon_2(n)$ to be the largest exponent of 2 that divides n. We start with an important lemma that will be referenced many times throughout the argument. This result is well-known and has a straightforward proof.

Lemma 1. *If* $\upsilon_2(a) \neq \upsilon_2(b)$, *then* $\upsilon_2(a+b) = \upsilon_2(a-b) = \min(\upsilon_2(a), \upsilon_2(b))$.

Here's another simple lemma that can be verified directly but is useful later.

Lemma 2. *If* $a > b > c$, *then* $ab - c > ac - b > bc - a$. *If some two of* $ab - c, ac - b, bc - a$ *are equal, then some two of* a, b, c *are equal.*

Now we proceed to solving the case where some two of the variables are equal.

Lemma 3. *If at least two of* a, b, c *are equal, the only solutions are* $(2, 2, 2)$ *and* $(2, 2, 3)$ *and their permutations.*

Proof. Without loss of generality, let $a = b$. Then we need the quantities $a^2 - c$ and $ac - a = a(c-1)$ to be powers of 2. Since a, b, c are all at least 2, $c - 1$ and a should both be powers of 2. Therefore, we can set $a = 2^x$ and $c = 2^y + 1$ for nonnegative integers x, y. Now, $a^2 - c = 2^{2x} - 2^y - 1$ must be a power of 2. Clearly $x > 0$. If $y > 0$, then this last quantity would be odd, so it would equal 1. Setting $2^{2x} - 2^y - 1 = 1$, we get $2^{2x-1} - 2^{y-1} = 1$. The only powers of 2 that differ by one are 1 and 2. Therefore, $x = 1, y = 1$. This leads to the solution $(2, 2, 3)$. Otherwise, $y = 0$. Then we need $2^{2x} - 2$ to be a power of 2. Dividing by 2, $2^{2x-1} - 1$ must be a power of 2. Since this quantity is odd, we need $x = 1$. Since $x = 1, y = 0$ we get the solution $(2, 2, 2)$. This exhausts all cases. ■

The next part contains four cases based on the parities of a, b, and c. We'll start with the simplest one first.

Lemma 4. *If two of* a, b, c *are odd and none are equal, we get no solutions.*

Proof. Without loss of generality, suppose a is even and b, c are odd. Since $ab - c$ and $ac - b$ are both odd and powers of 2, they both equal 1. Therefore, they are equal, contradicting Lemma 2. ■

Now we proceed to the case where all variables are even (and all unequal).

Lemma 5. *If* a, b, c *are even and no two are equal, we get no solutions.*

Proof. Without loss of generality, assume $\upsilon_2(a) \geq \upsilon_2(b) \geq \upsilon_2(c) \geq 1$. These quantities are all at least 1 because a, b, c are all even. Because $\upsilon_2(ab) = \upsilon_2(a) + \upsilon_2(b) > \upsilon_2(c)$, we have $\upsilon_2(ab - c) = \upsilon_2(c)$ by Lemma 1. Similarly, $\upsilon_2(ac - b) = \upsilon_2(b)$. Since $ab - c$ and $ac - b$ are both powers of 2, $ab - c = 2^{\upsilon_2(ab-c)} = 2^{\upsilon_2(c)} \leq c$ and $ac - b = 2^{\upsilon_2(ac-b)} = 2^{\upsilon_2(b)} \leq b$. Rearranging these gives $ab \leq 2c$ and $ac \leq 2b$. Multiplying these gives $a^2bc \leq 4bc \implies a \leq 2$. Since a is even, $a = 2$. This means that equality must hold in the previous inequalities. This means that $b \leq c$ and $c \leq b$, so $b = c$. But this reduces to Lemma 3. ■

We now continue to the case where a, b, c are all odd.

Lemma 6. *If a, b, c are all odd and no two are equal, we get the solution* $(3, 5, 7)$ *and its permutations.*

Proof. Without loss of generality, assume that $a > b > c > 1$. Since $ab - c > ac - b > bc - a$ by Lemma 2, set

$$ab - c = 2^x,$$
$$ac - b = 2^y,$$
$$bc - a = 2^z,$$

where $x > y > z$. From the first equation, $c = ab - 2^x$. Substituting this into the second equation, $a(ab - 2^x) - b = 2^y$. Taking this modulo 2^y, we get that $b(a^2 - 1) \equiv 0 \pmod{2^y} \implies 2^y \mid (a^2 - 1) = (a - 1)(a + 1)$. (Remember that $x > y$ and b is odd.) Because $\gcd(a - 1, a + 1) = 2$, $2^{y-1} \mid (a - 1)$ or $2^{y-1} \mid (a + 1)$. Regardless, $a \geq 2^{y-1} - 1$. If $a \geq 2^{y-1} + 1$, solving the second equation for c yields that

$$c = \frac{2^y + b}{a} < \frac{2^y}{a} + 1 \leq \frac{2^y}{2^{y-1} + 1} + 1 < 3,$$

a contradiction because c is odd and greater than 1. Therefore, $a = 2^{y-1} - 1$. Going back to the second equation $ac - b = 2^y$, we can solve for c to get that

$$c = \frac{2^y + b}{a} < \frac{2^y}{a} + 1 = \frac{2^y}{2^{y-1} - 1} + 1 \leq 5.$$

This follows because $y > 1$ and $b < a$, and one can easily verify that $\frac{2^y}{2^{y-1}-1} = 2 + \frac{2}{2^{y-1}-1} \leq 4$. Since $c > 1$ also, we need $c = 3$. Then $b = ac - 2^y = 3(2^{y-1} - 1) - 2^y = 2^{y-1} - 3$. Finally, because $ab - c$ is a power of 2, $(2^{y-1} - 1)(2^{y-1} - 3) - 3 = 2^{y+1}(2^{y-3} - 1)$ is a power of 2. Since the quantity in the parentheses is odd but also a power of 2, it must equal 1. Therefore, $y = 4$. This means that $a = 7, b = 5, c = 3$, giving the solution triple $(3, 5, 7)$ and its permutations. ■

Now we do the final case, where exactly one variable is odd.

Lemma 7. *If exactly one of a, b, c is odd and no two are equal, we get the solution* $(2, 6, 11)$ *and its permutations.*

Proof. Without loss of generality, let a be odd, and b, c be even. Since $bc - a$ is odd and a power of 2, we must have $bc - a = 1$. Therefore, $a = bc - 1$. It is easy to check that $bc - 1 > b, c$. Therefore, we can assume without loss of generality that $a > b > c$. By Lemma 2, $ab - c > ac - b$. Therefore, we can find integers x, y such that $ab - c = 2^x$ and $ac - b = 2^y$, and $x > y$. Now substitute $a = bc - 1$ to get the equations

$$b^2c - b - c = 2^x,$$
$$bc^2 - b - c = 2^y.$$

Adding these equations gives that $(bc - 2)(b + c) = 2^x + 2^y$. Therefore, $\upsilon_2((bc - 2)(b + c)) = \upsilon_2(2^x + 2^y) = y$. Since b, c are both even, $bc - 2 \equiv 2 \pmod 4 \implies \upsilon_2(bc - 2) = 1$. Therefore, $\upsilon_2(b + c) = y - \upsilon_2(bc - 2) = y - 1$, and so $b + c \geq 2^{y-1}$. To finish the bounding, we need one more inequality:

If $b, c \geq 2$, *then* $bc^2 \geq 4(b+c) - 8$.

(This rearranges to $(c-2)(bc+2b-4) \geq 0$, which is obvious given that $b, c \geq 2$.)

Returning to the proof, looking at the equation $bc^2 - b - c = 2^y$, because $b, c \geq 2$, we can apply this inequality to see that

$$\begin{aligned} 2^y &= bc^2 - b - c \geq 4(b+c) - 8 - (b+c) \\ &= 3(b+c) - 8 \geq 3 \cdot 2^{y-1} - 8 \implies 2^{y-1} \leq 8. \end{aligned}$$

Also from the previous inequalities,

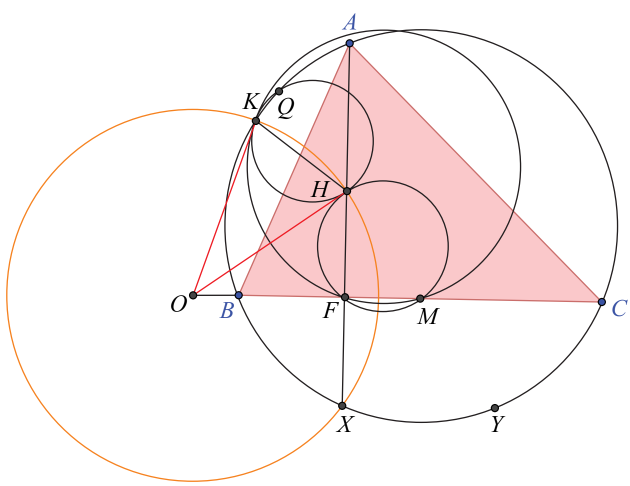
$$3(b+c) - 8 \leq 2^y \leq 16 \implies b + c \leq 8.$$

Since b, c are both even, it suffices to check the pairs $(b, c) = (2, 4), (2, 6)$ since they are not equal. $(2, 4)$ fails because $b^2c - b - c = 10$, not a power of 2, but $(2, 6)$ works. This means that $a = 11$. This leads to our final triple of $(2, 6, 11)$ and its permutations. ∎

In conclusion, the four working solutions are (2, 2, 2), (2, 2, 3), (3, 5, 7), (2, 6, 11) and permutations. It is easy to verify that these all work.

This problem was proposed by Serbia.

3. (Solution written by Michael Kural.)

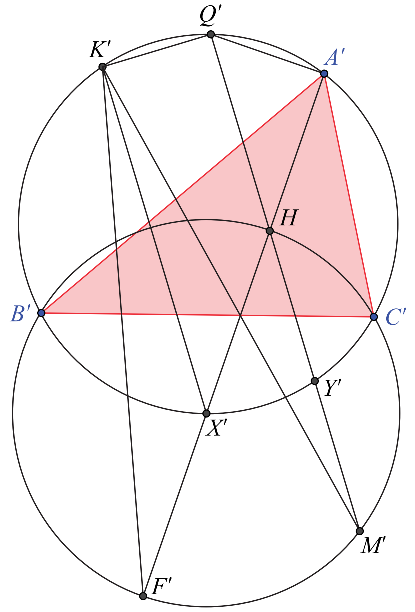


We provide two alternative solutions to showcase different interesting approaches. The first one uses the technique of geometrical inversion.

First solution. Let X, Y be the reflections of H across F, M, respectively. It is well-known that X and Y lie on Γ and furthermore that Y is the antipode of A on Γ. Now if YH intersects Γ again at Q^*, then

$$\angle AQ^*H = \angle AQ^*Y = 90^\circ$$

implying $Q^* = Q$. Thus, Q, H, M, and Y are collinear.



Now we consider an inversion about H with arbitrary radius. We denote the image of a point P under this inversion by marking it with a "prime" mark, e.g., P'. Note that $\angle HBA = \angle HCA$ implies $\angle HA'B' = \angle HA'C'$, so H is, by symmetry, the incenter of $\triangle A'B'C'$. Q' is the point of Γ' such that $\angle Q'A'H = 90^\circ$, but indeed this implies Q' must be the midpoint of major arc $B'A'C'$. K' is a point on Γ' such that $\angle K'Q'H = 90^\circ$. Denote by ω the circle through B', H, and C'; then $A'H$ and $Q'H$ meet ω at F' and M', respectively. The same lines $A'H$ and $Q'H$ also meet Γ' at X', Y', respectively. As $A'H$ is an angle bisector of $\angle B'A'C$, X' is the midpoint of minor arc BC. It is well-known that X' is the center of ω, which passes through B', C', H, and the A'-excenter of $\triangle ABC$, so F' is this excenter, and ω has diameter HF'.

Now it suffices to show that the circumcircle of $\triangle K'F'M'$ is tangent to line $K'Q'$. Because HF' is a diameter of ω, we know that $HM' \perp F'M'$. But $K'Q' \perp HM'$ as well (since Q', H, and M' are collinear). Also, $K'Q' \perp K'X'$ as $Q'X'$ is a diameter of Γ', but X' is the midpoint of HF'. This yields $F'K' = M'K'$, which along with $K'Q' \parallel F'M'$ implies the desired result. ■

Second solution. Let X and Y be defined as in the first solution. Let ω be the circumcircle of K, H, and X, and let O be its circumcenter. Note that O lies on BC and also that

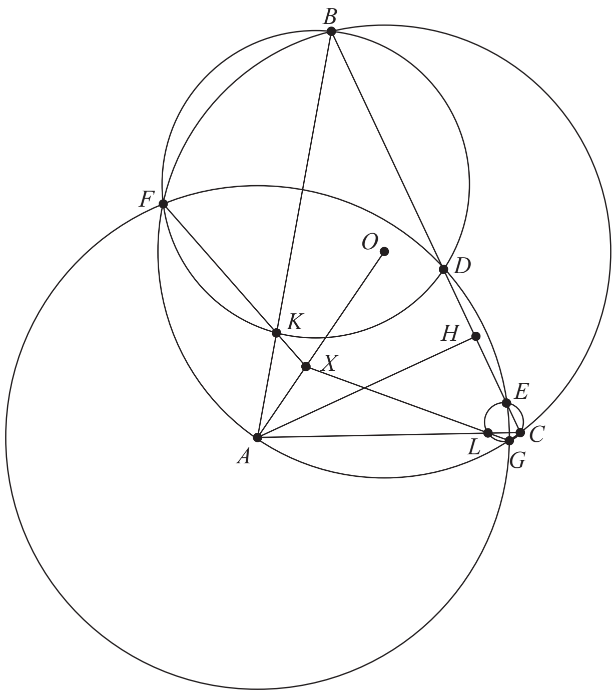
$$\angle KHQ = 90^\circ - \angle KQH = 90^\circ - \angle KQY = \angle KXA = \angle KXH$$

where the third equality follows since Y is the antipode of A on Γ. Thus, HM is tangent to ω, implying $\angle MHO = 90^\circ$.

Now note that the circumcircles of $\triangle HFM$ and $\triangle HKQ$ are tangent since they have diameters HM and HQ, respectively. Additionally, since $\angle MHO = 90^\circ$, O lies on the common tangent to the two circles. But O also lies on line MF (the perpendicular bisector of HX), which is the radical axis of the circumcircles of $\triangle KFM$ and $\triangle HFM$. Therefore, O is the radical center the circumcircles of $\triangle KFM$, $\triangle HFM$, and $\triangle HKQ$. Then $OH = OK$ implies OK is tangent to the circumcircle of $\triangle KHQ$, but OK is the radical axis of the circumcircles of $\triangle KFM$ and $\triangle KHQ$, so they are both tangent to OK at K, yielding the desired result. ■

This problem was proposed by Ukraine.

4. (Solution written by Allen Liu.)



Let H be the foot of the altitude from A to BC. Note that $AF = AG$ and $OF = OG$, so F and G are symmetric across AO. Hence, it suffices to show that

$$\angle AFK = \angle AGL.$$

Now $\angle AFK = \angle AFD - \angle KFD = \frac{180-\angle FAD}{2} - \angle KBD = \frac{90-\angle FAD}{2} - \angle B$ (since $AF = AD$ and $KFBD$ is cyclic).

Similarly, $\angle AGL = \frac{180-\angle GAE}{2} - \angle C$.

Therefore, it suffices to show that

$$\angle FAD - \angle GAE = 2(\angle C - \angle B).$$

We see

$$\begin{aligned}\angle FAD - \angle GAE &= \angle FAE - \angle GAD \\ &= (\angle FAO + \angle EAO) - (\angle GAO - \angle DAO) \\ &= \angle EAO + \angle DAO.\end{aligned}$$

But since $AD = AE$, D and E are symmetric about H, and we get $\angle EAO + \angle DAO = 2\angle HAO$.

Now

$$\angle HAO = \angle OAC - \angle HAC = (90 - \angle B) - (90 - \angle C) = \angle C - \angle B,$$

so we are done since we have that $\angle FAD - \angle GAE = 2(\angle C - \angle B)$, which is what we set out to prove. ■

This problem was proposed by Greece.

5. (Solution written by David Stoner.)

Let $P(x, y)$ denote the assertion $f(x + f(x + y)) + f(xy) = x + f(x + y) + yf(x)$. Then:

$$P(x, 1) \implies f(x + f(x + 1)) = x + f(x + 1) \quad \forall x \in \mathbb{R} \tag{1}$$

$$P(0, y) \implies f(f(y)) + f(0) = f(y) + yf(0) \quad \forall y \in \mathbb{R}. \tag{2}$$

If $f(x + 1) + x$ is equal to 1 for all values of x, then $f(x) = 2 - x$ for all x, which is indeed a solution.

Otherwise, there exists some x_0 such that $x_0 + f(x_0 + 1) = k$, and $k \neq 1$. According to (1) with $x = x_0$, it follows that $f(k) = k$. Then, putting $y = k$ into (2) gives $kf(0) = f(0)$, so $f(0) = 0$.

Using this fact:

$$P(x, -x) \implies f(-x^2) + f(x) = x - xf(x) \quad \forall x \in \mathbb{R} \tag{3}$$

$$P(-x, x) \implies f(-x^2) + f(-x) = -x + xf(-x) \quad \forall x \in \mathbb{R}. \tag{4}$$

Putting (3) and (4) together:

$$x - xf(x) - f(x) = f(-x^2) = -x + xf(-x) - f(-x) \quad \forall x \in \mathbb{R} \tag{5}$$

$$\implies 2x = (x + 1)f(x) + (x - 1)f(-x) \quad \forall x \in \mathbb{R}. \tag{6}$$

Lemma 8. *For each $a \in \mathbb{R}$, the following statements are equivalent:*

(i) $f(a) = a$,

(ii) $f(-a) = -a$,

(iii) $f(a) + f(-a) = 0$.

Proof. First, we prove the statements are all true when $a \in \{-1, 0, 1\}$. When $a = 0$, they are all true because $f(0) = 0$. Now plugging in $x = 1$ to (4) gives $2f(-1) = -1 + f(-1)$, so $f(-1) = -1$. Furthermore, plugging $x = 1$ into (3) gives $2f(1) = 2$, so $f(1) = 1$. It follows that all three statements hold for $a \in \{-1, 0, 1\}$.

Now assume that $a \notin \{-1, 0, 1\}$. Then setting $x = a$ in (6) gives:

$$2a = (a + 1)f(a) + (a - 1)f(-a) \tag{7}$$

$$\Leftrightarrow 0 = (a + 1)(f(a) - a) + (a - 1)(f(-a) + a). \tag{8}$$

Since $a + 1$ and $a - 1$ are nonzero, it follows that $f(a) = a$ iff $f(-a) = -a$, and therefore (i) and (ii) are equivalent.

Now if (i) is true, then (ii) is true from the above, and it follows that (iii) is true as well, so (i) $\implies$ (iii). Now assume that (iii) holds. Then (7) can be rewritten as:

$$2a = (a + 1)f(a) + (1 - a)f(a) \tag{9}$$

$$\implies 2a = 2f(a). \tag{10}$$

It follows that (i) is true as well, so (iii) $\implies$ (i). We have shown that (i) $\Leftrightarrow$ (ii) and (i) $\Leftrightarrow$ (iii), so all three statements are equivalent as required. ∎

Now let r be an arbitrary real number, and let $f(r) = s$. From $P(r, 0)$, it follows that $f(r + s) = r + s$. Furthermore, from $y = r$ in (2), it follows that $f(s) = s$, and so $f(-r - s) = -r - s$ and $f(-s) = -s$ from the lemma.

Now we note that by $P(-r, 2r+s)$:

$$f(-r+f(r+s))+f(-r(2r+s)) = -r+f(r+s)+(2r+s)f(-r) \quad (11)$$
$$f(s)+f(-r(2r+s)) = s+(2r+s)f(-r) \quad (12)$$
$$f(-r(2r+s)) = (2r+s)f(-r) \quad (13)$$

and by $P(r, -2r-s)$:

$$f(r+f(-r-s))+f(-r(2r+s)) = r+f(-r-s)-(2r+s)f(r) \quad (14)$$
$$f(-s)+f(-r(2r+s)) = -s+(2r+s)f(-r) \quad (15)$$
$$f(-r(2r+s)) = -(2r+s)f(r). \quad (16)$$

Combining (13) and (16), it follows that $(2r+s)f(-r) = -(2r+s)f(r)$, so $(2r+s)[(f(r)+f(-r)] = 0$. If $2r+s=0$, then $s=-2r$. Then $-r-s=r$, so $f(-r-s) = -r-s \implies f(r)=r$.

If $2r+s \neq 0$, then $f(r) = -f(-r)$, and from the lemma, this implies $f(r) = r$. It follows that $f(r)=r$ for all $r \in \mathbb{R}$, and this is indeed a solution.

In conclusion, the two solutions to the original functional equation are the linear functions $f(x) = 2-x$ and $f(x)=x$, and we are done. ■

This problem was proposed by Albania.

6. (Solution written by Ryan Alweiss.)

Let $f(j) = j + a_j$, so f is injective with $x < f(x) \leq x + 2015$. We claim that at most 2015 integers are not in the range of f. Assume not, so for some k we have 2016 of $A = \{1, 2, \ldots, k\}$ are not in the range. Note then that $f(A)$, the image of A, is a subset of $\{1, 2, \ldots, k+2015\}$ but excludes at least 2016 of $\{1, 2, \ldots, k\}$. Thus, it has cardinality $k-1$, contradicting injectivity. We get that in fact at most 2015 positive integers are not in the range, as claimed.

Because f grows slowly, the structure of the function should not be too hard to describe. Intuitively, everything should be obtainable uniquely by applying f to some small term. This intuition can in fact be made rigorous. First, assign all integers not in the range of f (there are at most 2015, as shown above) distinct colors. Now, at any moment consider the smallest uncolored r. We have r is in the range, so we give it the same color as $f^{-1}(r)$. In this way, we color the positive integers with some number of colors. Each color represents a "chain" of subsequently applying f.

All of the above have helped to understand the function, and a bit of unrigorous thinking motivates the final steps. If we let b be the number of these "chains," we see that each one has "density" $\frac{1}{b}$, and so the jumps a_j between subsequent elements in them are "about" b.

With a solid understanding of the function, it suffices to begin to examine the inequality. First, we will relax the statement and prove for $n - m \geq 2015$, where n and m are both sufficiently large, that the inequality is true. We pick m, n large enough so that all colors appear in the set $K = m+1, \ldots, n$; if a color did not appear in this set, the gap between consecutive elements of that color would be at least $(n+1) - m > 2015$, a contradiction. For each color c, denote by M_c the minimal element of color c greater than m, whereas N_c is the minimal element of that color greater than n. Let X be the set of M_c and let Y be the set of N_c. Now, when we apply f to $K = \{m+1, \ldots, n\}$, the result is K except with X replaced by Y because for each color c the numbers of that color in $f(K)$ are those in K except with M_c replaced by N_c. For each color, the difference of the sum of the terms of that color in $f(K)$ and those in K is $(N_c - M_c) = ((N_c - n) - (M_c - m)) + (n - m)$. Thus,

$$\begin{aligned}\sum_{m+1}^{n} a_j &= \sum_{k \in K} f(k) - k \\ &= \sum_{c} b + (N_c - n) - (M_c - m) \\ &= b(n-m) + \sum_{c} (N_c - n) - (M_c - m).\end{aligned}$$

So the quantity whose absolute value is considered in the problem, which is $\sum a_j - b(n - m)$, is just $\sum_c (N_c - n) - (M_c - m)$. Both the $N_c - n$ and $M_c - m$ are sets of b distinct integers, and since N_c and M_c were defined to be the minimal elements of each color bigger than n and m, respectively, they must hit $n + 1$ and $m + 1$, respectively. Hence, the number 1 must show up in each. Thus, each of these is between $1 + 2 + \cdots + b$ and $1 + (2015) + \cdots + (2017 - b)$, so their difference has magnitude at most $(2015 - b)(b - 1) \leq 1007^2$, finishing the problem in this case.

It remains to check the case $n - m < 2015$, but fortunately this is not too difficult. Let A_j denote $a_j - b$. Assume that for some fixed n' and m' both large we have, without loss of generality, that $A_{m'+1} + \cdots + A_{n'} < -1007^2$. Then for all $m \geq 2015 + n'$, by the above, we have that $A_{m'+1} + \cdots + A_m$ and $A_{n'+1} + \cdots + A_m$ are between -1007^2 and 1007^2. As the latter is greater than 1007^2 more than the former, it is positive. So if $n > m > 9000 + n'$ we have that $A_{n'+1} + \cdots + A_m$ and $A_{n'+1} + \cdots + A_n$, in addition to being at most 1007^2, are positive. So the difference $A_{m+1} + \cdots + A_n$ has magnitude at most 1007^2, and we're done. ■

This problem was proposed by Australia.

Results. The IMO was held in Chiang Mai, Thailand, on July 10–11, 2015. There were 577 competitors from 104 countries and regions. On each day, contestants were given four and a half hours for three problems.

The top score of 42/42 was achieved by a single student, Zhuo Qun (Alex) Song, from the team representing Canada. This student was also a multitime USAMO winner and attended Phillips Exeter Academy in New Hampshire, as well as the USA Math Olympiad Summer Program.

The USA team won five gold medals and one silver medal, earning a total of 185 points, placing first in the world. The second-place team (China) earned a total of 181 points, with four gold and two silver medals. The previous USA win was 21 years prior, in 1994. This historic victory was covered in media outlets spanning the Washington Post, National Public Radio, the Los Angeles Times, late night comedy, and international newspapers such as the Beijing News and the Straits Times. Po-Shen Loh served as the team leader (national coach), and John Berman served as the deputy team leader.

The students' individual results were as follow.

- Ryan Alweiss, who finished 12$^{\text{th}}$ grade at Bergen County Academies in Closter, NJ, won a gold medal.
- Michael Kural, who finished 11$^{\text{th}}$ grade at Greenwich High School in Riverside, CT, won a silver medal.

- Allen Liu, who finished 11th grade at Penfield Senior High School in Penfield, NY, won a gold medal.
- Yang Liu, who finished 12th grade at Ladue Horton Watkins High School in St. Louis, MO, won a gold medal.
- Shyam Narayanan, who finished 12th grade at Blue Valley West High School in Overland Park, KS, won a gold medal.
- David Stoner, who finished 12th grade at South Aiken High School in Aiken, SC, won a gold medal.

Carl B. Allendoerfer Awards

The Carl B. Allendoerfer Awards, established in 1976, are made to authors of articles of expository excellence published in *Mathematics Magazine*. The Awards are named for Carl B. Allendoerfer, a distinguished mathematician at the University of Washington and president of the Mathematical Association of America, 1959–1960.

Daniel Heath

"Straightedge and Compass Constructions in Spherical Geometry," *Mathematics Magazine*, Volume 87, Number 5, December 2014, pages 350–359.

Mathematicians and most mathematics students are at least somewhat familiar with the Euclidean construction "game": deciding what is and is not constructible using only a straightedge and compass. Historical high points include Euclid's construction of a regular pentagon, Gauss's proof that a regular 17-gon is constructible and Wanzel's complete classification of those numbers n for which regular n-gons are and are not constructible.

In this well-written article, Heath introduces us to the *spherical* construction game. Beginning with a set of starting conditions (the *standard* starting conditions assume the distance between two given fixed points A and B is $\pi/4$) and working on the unit sphere S^2, the goal is to decide which points on the sphere are constructible using only a (spherical) straightedge and compass. Heath begins with examples that illustrate how the starting conditions affect the set of constructible points then develops his main results in six theorems. Perhaps the most remarkable of the theorems is that, assuming standard starting conditions, the set of constructible points is dense in S^2. The author extends his results to appropriately defined constructions in the projective plane and concludes that the set of constructible points is dense in the projective plane as well. The paper concludes with some open questions, including whether the constructibility question for the sphere is answerable in general.

Not only are these results interesting and intriguing, but an essential charm of the paper lies in the clear organization, the engaging writing style, and the broad array of mathematical methods and areas employed: classical Euclidean geometry, analytic geometry, group structures, and the use of both geometric and analytic arguments in the proofs. This fine paper invites multiple readings.

Response from Daniel Heath

I have just been informed that I have won a Carl B. Allendoerfer prize for expository excellence. First, let me say that I am surprised and humbled; I have never considered myself an excellent expositor and think I must come up quite short when compared to great math writers such as the namesake of this award.

I'd like to thank Abigail Thompson, Tsuyoshi Kobayashi, and Celine Dorner for the great contributions they made to my career and my wife, Yumiko Muraoka, for her

Math. Mag. **88** (2015) 316–319. doi:10.4169/math.mag.88.4.316.

constant love and encouragement. Lastly, I'd like to extend thanks to the editor and referee for great suggestions, and the committee for choosing my work for this honor. There are many great expository works published in *Mathematics Magazine*, so I think the job of choosing only one or two to honor is very difficult.

Since I have your attention, consider $p : S2\backslash\{\text{south pole}\} \rightarrow \mathbb{R}2$ defined by

$$p(x, y, z) = \left(\frac{x}{1+z}, \frac{y}{1+z}\right) = (X, Y).$$

Horribly abusing notation, we obtain

$$p(ax + by + cz + d) = \left\{\begin{matrix} \left(X - \frac{a}{c+d}\right)^2 + \left(Y - \frac{b}{c+d}\right)^2 = \frac{1-d^2}{(c+d)^2} & \textit{if } c \neq d \\ aX + bY + c = 0 & \textit{if } c = d \end{matrix}\right\},$$

So that spherical circles and geodesics are taken to Euclidean circles or geodesics. I've reached my word limit, so you'll have to figure out what this means yourself.

Biographical Note

Daniel J. Heath, also known by his nickname "deej," graduated from St. Olaf College and was in one of the first few groups to participate in the Budapest Semesters in Mathematics. He earned his Ph.D. at the University of California at Davis under the direction of Abigail Thompson and went on to do postdoctoral work at Nara Women's University with then mentor and now friend Tsuyoshi Kobayashi. It was there that he met his wife, Yumiko Muraoka.

Heath works at Pacific Lutheran University, where he is an associate professor of mathematics and the chair of the department. His research interests range from algebra to mathematical origami to low-dimensional topology. In his free time he loves mushroom hunting with his five-year-old son, King; playing jazz steelpan; bicycling; and woodworking. He plans to use the prize money to fund a trip to Trinidad, where he will deliver a suitcase full of microscopes to the Cascade School for the Deaf.

Andrew Beveridge and Stan Wagon

"The Sorting Hat Goes to College," *Mathematics Magazine*, Volume 87, Number 4, October 2014, pages 243–251.

In "The Sorting Hat Goes to College," Beveridge and Wagon recall for us the scene in which Harry Potter and his entering class at Hogwarts meet the Sorting Hat, which assigns each student to one of the four houses at Hogwarts, influencing their destiny. The authors note that there is a pattern in the assignments: Each year, the incoming students are split equally among the houses and apportioned evenly by gender. This serves as an introduction to the analogous problem, very real in many colleges and universities, of matching incoming students with first-year seminars. When first-year seminar offerings span a wide variety of disciplines, the task of assigning incoming students according to their preferences is challenging and labor intensive. Beveridge and Wagon recognize it as a constrained optimization problem, and in this paper, they show how they solved it with mathematics.

The authors trace the evolution of their algorithm. In the first phase, in 2008–2009, senior Sean Cooke took on this project in his combinatorial optimization class at Macalester College. Cooke and Beveridge began with the classic minimum weight perfect matching algorithm (the "Hungarian algorithm") of graph theory. Following an essential practice of all good consultants, Cooke and Beveridge refined the algorithm over the course of several meetings with the Office of Academic Programs, using their feedback to better understand and implement their priorities. The resulting algorithm, which they called the "Hungarian Hat," was successfully adapted by Macalester College. Compared to the former manual procedures, it placed 4% more students in their first or second choices and placed far fewer students in their fourth choices.

Five years later, the "Hungarian Hat" algorithm underwent a significant revision; in fact, it was replaced by an algorithm using integer linear programming (ILP). The authors describe the shortcomings of the graph theoretic approach—mainly an inherent rigidity—that led them to the linear programming method. They describe how they incorporate the constraints (course size, gender balance, international balance) and how they add some overriding flexibility to the objectives. The authors offer a helpful measure of complexity: In the ILP approach, the number of variables is $4n + 22m$ where n is the number of students and m is the number of courses. Using Mathematica's ILP package, Macalester's cases run in under 5 seconds. However, the authors note that ILP is an NP-complete problem so that, if the number of students was in the thousands, the benefits of ILP might be outweighed by the added computational time.

The writing is lively and entertaining, and the mathematics is accessible. The particular problem is very real to students, and it vividly illustrates the use of both graph theory and ILP to solve real-world problems.

Response from Andrew Beveridge and Stan Wagon

We are elated that the MAA has honored our paper with a Carl B. Allendoerfer Award. This has been a particularly fulfilling project for us, for it has brought together the myriad roles of our professional lives. This work started as a senior capstone project with a student, inspired by a classroom discussion on the applications of graph theory. It created a valuable, ongoing collaboration with staff and administration. Even now, this project continues to have a direct impact. Our Sorting Hat improves the staff's work life by replacing a burdensome manual task with a fluid and responsive tool. The incoming students are placed in their preferred courses at a rate that cannot be improved. We continue to feature this project in the classroom, where the optimization problem resonates strongly with our students, sparking their interest in modeling and optimization techniques. Faculty members strive for excellence in teaching their courses, generating individual projects for students, and serving both the college and the wider professional community. Projects that combine all of these are rare, but this one does it all. It is gratifying to see our methods used outside of our own college and deeply satisfying to receive an award that underscores the value and appeal of useful applications of mathematics.

Biographical Notes

Andrew Beveridge received a BA in mathematics from Williams College in 1991 and a Ph.D. in mathematics from Yale University in 1997. Bookended by visiting positions at Carnegie Mellon University (1997–1998 and 2005–2007), he worked as a Silicon Valley software developer in the heady (and not-so-heady) days of the dot-com boom and bust. He joined the faculty at Macalester College in 2007. His research interests span combinatorics and computer science, including random walks, random graphs, and pursuit–evasion games. He is an avid volleyball player, plays bass guitar in a band called "Math Emergency," and owns the world's only velvet painting of Erdős.

Stan Wagon, now retired, has taught at Smith and Macalester Colleges. He enjoys writing about interesting mathematics and has won four MAA writing awards. His main current interests are the use of Mathematica in various aspects of research and applications and the completion of a second edition of his 1985 book, *The Banach–Tarski Paradox*. He is the founding editor of *Ultrarunning Magazine* but now prefers to cover long distances on skis and snow rather than dry ground. He has won several awards for geometrical snow sculpture at an international competition but is most famous for the design and construction of a bicycle with square wheels that rolls smoothly on a road made of catenaries.

So, You Think You Have Problems?

In 1745, Tobias Mayer together with colleagues, Johann Baumgarten and Johann Pinitz, published the reference Mathematischer Atlas. In the above frontispiece, the goddess of mathematics, holding a compass in her right hand, sits at the center of the illustration surrounded by various mathematical paraphernalia. She is besieged by three supplicants: the Muse of Geometry, denoted by her crown; Hermes, Messenger of the Gods; and an obviously perplexed astronomer. Geometry, on the left, gestures to the distant construction site—something is not right, Hermes holds a map, and, finally, the astronomer points to the heavens, the alignments are off. Mathematics listens patiently to these problems.

This image, together with over a thousand other mathematically related items and illustrations are available for reference and teaching purposes in the archive "Mathematical Treasures" contained in Convergence at the MAA website.

—Frank Swetz, Pennsylvania State University

You are invited

MAA Panel Discussion at JMM 2016

What Belongs in a 21st-Century Geometry Course?

Wednesday, January 6, 2016

9:30 AM – 10:50 AM

Washington State Convention Center

Room 619

Panelists:

Matthew Harvey, *The University of Virginia's College at Wise*

Thomas Q. Sibley, *St. John's University*

Gerard Venema, *Calvin College*

Any teacher of an undergraduate geometry course has to make difficult choices about what topics to include and exclude. She will be faced with decisions about the role of technology and newer pedagogies. The needs of future high-school teachers, the demands of the Common Core, and the recommendations of the MAA's CUPM course report must be taken into account. Please join us for a discussion of balancing all these priorities led by textbook authors Tom Sibley, Matthew Harvey, and Gerard Venema.

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